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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

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# EMBEDDING THEOREMS FOR FUNCTIONS DEFINED ON UNBOUNDED DOMAINS

*(Presented by Academician I. M. Vinogradov on 24 VI 1963)*

We shall consider functions defined, together with their derivatives up to order  $r$ , on an unbounded domain of Euclidean space. We shall assume that the derivatives of order  $r$  of these functions are not summable to the power  $p$  ( $1 < p < \infty$ ) on the indicated domain, but are summable to the power  $p$  with a certain weight having a power rate of tending to zero at infinity. For the indicated class of functions we shall obtain a number of embedding theorems, both direct and inverse. For the usual function spaces (i.e., spaces of functions all of whose derivatives of order  $r$  are summable to the power  $p$  on the domain), as corollaries of these theorems one obtains new embedding theorems reflecting the specific nature of unbounded domains, connected with the fact that in this case the summability of derivatives to the power  $p$  on the domain does not imply the summability of the function itself (as is the case for bounded domains).

Let us introduce notation. Let  $B$  be a Banach space,  $u \in B$ ; by  $|u, B|$  we shall denote the norm of the element  $u$  in the space  $B$ . Let  $E^n$  be the  $n$ -dimensional Euclidean space of points  $x = (x_1, \dots, x_n)$ ;  $G$  a domain in  $E^n$ ;  $\Gamma$  its boundary;  $u = u(x)$  a function defined on  $G$ ; by  $u|_\Gamma$  we shall denote the trace of the function  $u$  on  $\Gamma$ . For convenience we shall also call the function itself its trace. Let  $B_1$  be some Banach space of functions defined on  $G$ , and  $B_2$  a Banach space of functions defined on  $\Gamma$  (or also on  $G$ ). If for every function  $u \in B_1$  there exists its trace, if this trace belongs to the space  $B_2$ , and if the operator assigning to a function  $u \in B_1$  its trace is bounded, then it is called an **embedding operator**, is written  $B_1 \rightarrow B_2$ , and one says that the space  $B_1$  is embedded in the space  $B_2$ . If, however, there exists a bounded operator assigning to each function  $\varphi \in B_2$  such a function  $u \in B_1$  that  $\varphi$  is the trace of  $u$ , then this operator is called an **inverse embedding operator** and is written  $B_2 \rightarrow B_1$ .

For simplicity we shall formulate our theorems only for the case of the space and the half-space; in an analogous way they are formulated also for unbounded domains with sufficiently smooth  $(n - 1)$ -dimensional boundary.

Let

$$E^+ = \{x : x_n > 0\}; \quad E^{n-1} = \{x : x_n = 0\}; \quad \rho = \sqrt{x_1^2 + \dots + x_n^2};$$

$|x-y|$  be the distance between points  $x \in E^n$  and  $y \in E^n$ ;  $Q^n$  the  $n$ -dimensional ball of unit radius with center at the origin;  $Q^{\dagger n} = E^n \cap Q^n$ ;  $1 \leq p \leq \infty$ ,  $\alpha \geq 0$ ;  $r$  a nonnegative integer;  $u = u(x)$  a function defined on  $E^{\dagger n}$  together with its generalized derivatives up to order  $r$ , inclusive.

Denote by  $L_{p,\alpha}^{(r)}(E^{\dagger n})$  the linear normed space with norm

$$\left| u, L_{p,\alpha}^{(r)}(E^{\dagger n}) \right| = \sum_{r_1 + \dots + r_n = r} \left| \frac{1}{(1+\rho)^\alpha} \frac{\partial^r u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}}, L_p(E^{\dagger n}) \right|,$$

and by  $W_{p,\alpha}^{(r)}(\bar{E}^n)$  the linear normed space of functions with norm

$$\left| u, W_{p,\alpha}^{(r)}(\bar{E}^n) \right| = \left| u, L_{p,\alpha}^{(r)}(\bar{E}^n) \right| + \left| u, L_p(\bar{Q}^n) \right|$$

for  $r \geq 1$ , and with norm

$$\left| u, W_{p,\alpha}^{(0)}(\bar{E}^n) \right| = \left| u, L_{p,\alpha}^{(0)}(\bar{E}^n) \right|$$

for  $r = 0$ . The space  $L_{p,\alpha}^{(r)}(\bar{E}_1^n)$  is defined analogously for the "strip"  $\bar{E}_1^n \{x : 0 \leq x_n \leq 1\}$ .

For  $\alpha = 0$  the space  $L_{p,\alpha}^{(r)}$  becomes the well-known space  $L_p^{(r)}$  of S. L. Sobolev <sup>(1)</sup>.

Let now  $0 < \beta < 1$ . Put

$$I_{\alpha,\beta}(u) = \left\{ \int_e \frac{dE_x^n}{(1+\rho)^{\alpha p}} \int_{|x-y| \leq 1} \frac{|u(x) - u(y)|^p dE_y^n}{|x-y|^{\beta p + n}} \right\}^{1/p}$$

and denote by  $L_{p,\alpha}^{(r+\beta)}(E^n)$  the linear normed space of functions with norm

$$\left| u, L_{p,\alpha}^{(r+\beta)}(E^n) \right| = \sum_{r_1 + \dots + r_n = r} I_{\alpha,\beta} \left( \frac{\partial^r u}{\partial x_1^{r_1} \dots \partial x_n^{r_n}} \right), \quad r = 0, 1, \dots,$$

and by  $W_{p,\alpha}^{(r+\beta)}(E^n)$  the linear normed space of functions with norm

$$\left| u, W_{p,\alpha}^{(r+\beta)}(E^n) \right| = \left| u, L_{p,\alpha}^{(r+\beta)}(E^n) \right| + \left| u, L_p(Q^n) \right|.$$

We note that for any  $\gamma \geq 0$ , obviously, the embedding  $W_{p,\alpha}^{(\gamma)} \rightarrow L_{p,\alpha}^{(\gamma)}$  holds; on the other hand, if  $u \in L_{p,\alpha}^{(r)}(E^n)$ , then  $u \in W_{p,\alpha}^{(r)}$ , although the norm of the

function  $u$  in the space  $W_{p,\alpha}^{(r)}$  is not estimated by its norm in the space  $L_{p,\alpha}^{(r)}$  for  $r > 0$ .

**Theorem 1.** *The following embeddings hold: for  $\alpha > n/p - 1$ ,  $r \geq k \geq 0$  ( $r, k$  integers)*

$$W_{p,\alpha}^{(r)}(\bar{E}^n) \rightarrow W_{p,\alpha+k}^{(r-k)}(\bar{E}^n); \quad (1)$$

$$W_{p,\alpha}^{(r)}(\bar{E}^n) \rightarrow L_{p,\alpha+k}^{(r-k)}(\bar{E}^n);$$

$$W_{p,\alpha}^{(r)}(\bar{E}^n) \rightarrow W_{p,\alpha+1-1/p}^{(r-1)}(E^{n-1}), \quad (2)$$

and for any  $\alpha \geq 0$

$$L_{p,\alpha}^{(r)}(\bar{E}^n) \rightarrow L_{p,\alpha}^{(r-1/p)}(E^{n-1}); \quad (3)$$

$$W_{p,\alpha}^{(r)}(\bar{E}^n) \rightarrow L_{p,\alpha}^{(r-1/p)}(E^{n-1}).$$

**Corollary.** *For any  $\varepsilon > 0$  the embeddings hold:*

$$\bar{W}_2^{(r)}(\bar{E}^n) \rightarrow W_{2,n/2+k-1+\varepsilon}^{(r-k)}(\bar{E}^n);$$

$$W_2^{(r)}(\bar{E}^n) \rightarrow W_{2,(n-1+\varepsilon)/2}^{(r-1)}(E^{n-1}).$$

For  $\varepsilon = 0$  the indicated embeddings cease to be valid.

**Theorem 2.** *Let functions  $\varphi_k$  be specified on  $E^{n-1}$  such that*

$$\varphi_k \in L_{p,\alpha}^{(r-k-1/p)}(E^{n-1}), \quad k = 0, 1, \dots, r-1.$$

Then there exists a function  $u \in L_{p,\alpha}^{(r)}(\bar{E}_1^n)$  such that

$$\left. \frac{\partial^k u}{\partial x_n^k} \right|_{E^{n-1}} = \varphi_k, \quad k = 0, 1, \dots,$$

$$\left| u, L_{p,\alpha}^{(r)}(\bar{E}^n) \right| \leq \sum_{k=0}^{r-1} \left| \varphi_k, L_{p,\alpha}^{(r-k-1/p)}(E^{n-1}) \right|.$$

**Corollary.** For  $\alpha = 0$  the embedding

$$L_{p,0}^{(r-1/p)}(E^{n-1}) \rightarrow L_{p,0}^{(r)}(\vec{E}^n) \quad (4)$$

holds.

We note that the embeddings (3) and (4), as well as Theorem 2 for  $\alpha = 0$ , were proved earlier by V. A. Solonnikov and V. P. Il' in <sup>(2,5)</sup>. Embedding theorems for weighted spaces of functions growing at infinity were considered earlier in the works of A. S. Dzhafarov <sup>(3)</sup>; however, in the theorems obtained by him the weight did not change when the parameters  $r$  and  $n$  were changed. In our theorems it is precisely the relation between  $r$ ,  $n$ , and  $\alpha$  that is essential; clarifying it made it possible to obtain sufficiently sharp direct and inverse embedding theorems for the class of functions under consideration. Applications of the theorems obtained to the variational method for solving partial differential equations in unbounded domains are given in the author's paper <sup>(4)</sup>.

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*Note: Figure translations are in progress. See original paper for figures.*

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