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Abstract

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BOUNDARY ESTIMATES FOR THE SOLUTION OF THE THIRD BOUNDARY-VALUE PROBLEM FOR A PARABOLIC EQUATION

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For elliptic equations, $(2+\alpha)$ a priori estimates up to the boundary were obtained by Schauder ([1,2]) for the solution of the first boundary-value problem, by Fiorenza ([3]) for the boundary-value problem with oblique derivative, and by Agmon, Douglis, and Nirenberg ([4]) for boundary-value problems with general boundary conditions. In the theory of boundary-value problems for parabolic equations, up to the present time a priori estimates (up to the boundary) have been obtained only for solutions of the first boundary-value problem (see ([5,6])). In our note a $(2+\alpha)$ a priori estimate (up to the boundary) is established for the solution of the third boundary-value problem (with conormal derivative) for a parabolic equation of the second order and for certain parabolic systems.

§ 1. We consider the solution $u(x,t)$ of the following boundary-value problem:

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t), \quad (x,t)\in Q; \quad (1)$$

$$u(x,0) = \psi(x), \qquad x\in\Omega = \overline{Q}\cap\{t=0\}; \quad (2)$$

$$\frac{\partial u(x,t)}{\partial N} + b(x,t)u(x,t) = \varphi(x,t), \qquad (x,t)\in\Gamma, \quad (3)$$

where $\partial/\partial N$ is the derivative along the conormal to the surface Γ at the point (x,t). Q is a domain (possibly unbounded in the x_i) in the space $(x,t) = (x_1,\dots,x_n,t)$, lying between the hyperplanes $t=0$ and $t=T>0$ and bounded by the lateral surface Γ. By ∂Q we denote the "normal" boundary of Q, i.e. the set $\partial Q = \Gamma\cup\Omega$.

A_1. The lateral surface Γ has at each of its points a tangent plane nowhere orthogonal to the axis Ot. Moreover, for each point $P(x,t)\in\Gamma$ there exists an $(n+1)$-dimensional sphere $S_\delta(P)$ with center at P and radius $\delta>0$ (where δ

does not depend on the choice of the point P on Γ) such that the part of Γ lying in the sphere $S_\delta(P)$ can be represented, for some i $(1 \le i \le n)$, in the form

$$x_i = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n; t),$$

where the function h has derivatives with respect to x_k $(k = 1, \dots, i-1, i+1, \dots, n)$ up to the second order inclusive, satisfying the Hölder condition in x and t with exponent $\alpha > 0$, and a first derivative with respect to t, satisfying the Hölder condition in x and t with exponent $\alpha > 0$.

For what follows it is convenient to introduce the following norms. Let

$$d(P_1, P_2) = \left(\sum_{i=1}^{n} (x_i^{(1)} - x_i^{(2)})^2 + |t_1 - t_2| \right)^{1/2}$$

be the parabolic distance between the points $P_1(x^{(1)}, t_1)$ and $P_2(x^{(2)}, t_2)$. For

For a function $u(x,t) = u(P)$ defined on the set B, put

$$[u]_0^B = \sup_B |u(P)|, \qquad [u]_1^B = \sup_{i,P} \left| \frac{\partial u(P)}{\partial x_i} \right|,$$

$$[u]_2^B = \max \left\{ \sup_{i,j,P} \left| \frac{\partial^2 u(P)}{\partial x_i \partial x_j} \right|, \sup_P \left| \frac{\partial u(P)}{\partial t} \right| \right\},$$

$$[u]_\alpha^B = \sup_{P_1, P_2} (d^{-\alpha}(P_1, P_2) |u(P_1) - u(P_2)|),$$

$$[u]_{1+\alpha}^B = \max \left\{ \sup_{i, P_1, P_2} \left(d^{-\alpha}(P_1, P_2) \left| \frac{\partial u(P_1)}{\partial x_i} - \frac{\partial u(P_2)}{\partial x_i} \right| \right), \right.$$

$$\left. \sup_{P, P_0} \left(|t_1 - t_2|^{-\frac{1+\alpha}{2}} |u(P) - u(P_0)| \right) \right\},$$

where $P = P(x, t_1)$, $P_0 = P_0(x, t_2)$,

$$[u]_{2+\alpha}^B = \max \left\{ \sup_{i,j,P_1,P_2} \left(d^{-\alpha}(P_1, P_2) \left| \frac{\partial^2 u(P_1)}{\partial x_i \partial x_j} - \frac{\partial^2 u(P_2)}{\partial x_i \partial x_j} \right| \right), \right.$$

$$\sup_{P_1, P_2} \left(d^{-\alpha}(P_1, P_2) \left| \frac{\partial u(P_1)}{\partial t} - \frac{\partial u(P_2)}{\partial t} \right| \right),$$

$$\sup_{i,P,P_0}\left(|t_1-t_2|^{-\frac{1+\alpha}{2}}\left|\frac{\partial u(P)}{\partial x_i}-\frac{\partial u(P_0)}{\partial x_i}\right|\right)\Bigg\},$$

$$|u|_\alpha^B=[u]_0^B+[u]_\alpha^B,\qquad |u|_1^B=[u]_0^B+[u]_1^B,\qquad |u|_2^B=|u|_1^B+[u]_2^B,$$

$$|u|_{1+\alpha}^B=|u|_1^B+[u]_{1+\alpha}^B,\qquad |u|_{2+\alpha}^B=|u|_2^B+[u]_{2+\alpha}^B.$$

Concerning the coefficients of equation (1) and the functions entering into (2) and (3), we shall assume that the following conditions are satisfied:

A_2. Equation (1) is of parabolic type in $\overline{Q}$; the matrix $a_{ij}(x,t)$ is symmetric with a positive-definite quadratic form in $\overline{Q}$, and

$$\det|a_{ij}(x,t)|\geq a>0.$$

$$A_3.\quad |a_{ij}|_\alpha^{\overline{Q}}+|b_i|_\alpha^{\overline{Q}}+|c|_\alpha^{\overline{Q}}\leq M_1,\qquad |f|_\alpha^{\overline{Q}}<+\infty.$$

$$A_4.\quad |b|_{1+\alpha}^\Gamma\leq M_2,\qquad |\varphi|_{1+\alpha}^\Gamma<+\infty,\qquad |\psi|_{2+\alpha}^\Omega<+\infty.$$

A_5. The right-hand side f from (1), the initial function ψ from (2), and the boundary function φ from (3) are assumed to be compatible in virtue of equation (1), i.e. they satisfy equation (1) on $\Omega\cap\Gamma$.

§ 2. **Lemma 1** (cf. (1, 6)). *If $u(x,t)$ is a solution of the homogeneous parabolic equation with constant coefficients*

$$L^0u=\sum_{i,j=1}^n a_{ij}^0\frac{\partial^2 u}{\partial x_i\partial x_j}-\frac{\partial u}{\partial t}=0$$

(where $\det|a_{ij}^0|\geq a>0,\ \sup|a_{ij}^0|\leq M_1$) in the rectangle

$$V_R=\{(x,t),\ |x_i|<R;\ i=1,2,\dots,n;\ 0<t<R^2\},$$

continuous in $\overline{V}_R$, then

$$[u]_{2+\alpha}^{V_{R,\lambda}}\leq C(a,M_1)\lambda^{-(2+\alpha)}\sup_{\partial V_R}|u(P)|,$$

where

$$V_{R,\lambda}=\{(x,t),\ |x_i|\leq R-\lambda,\ i=1,2,\dots,n;\ \lambda^2\leq t\leq R^2\},\qquad 0<\lambda\leq 1.$$

Lemma 2 (basic). If $f(x,t)$ is defined in the quarter-space

$$D = \{(x,t),\ |x_i| < +\infty,\ i = 1,2,\dots,n-1,\ 0 < x_n < +\infty,\ 0 < t < +\infty\},$$

with $|f|_\alpha^D < +\infty$, the function $\psi(x)$ is defined in $\Omega_1 = \overline{D} \cap \{t = 0\}$, with $|\psi|_{2+\alpha}^{\Omega_1} < +\infty$, and $\varphi(x_1,\dots,x_{n-1};t)$ is defined on the hyperplane

$$\Gamma_1 = \overline{D} \cap \{x_n = 0\},$$

with $|\varphi|_{1+\alpha}^{\Gamma_1} < +\infty$, then there exists a function $v(x,t)$ such that

$$L^0 v = f(x,t), \qquad (x,t) \in D; \tag{4}$$

$$v(x,0) = \psi(x), \qquad x \in \Omega_1, \tag{5}$$

$$\frac{\partial v(x,t)}{\partial N_0} = \varphi(x,t), \qquad (x,t) \in \Gamma_1 \tag{6}$$

(where $\partial/\partial N_0$ is the derivative with respect to the conormal for equation (4) to the hyperplane Γ_1 at the point (x,t)), and

$$[v]_{2+\alpha}^D \le C(a, M_1)(|f|_\alpha^D + [\psi]_{2+\alpha}^{\Omega_1} + [\varphi]_{1+\alpha}^{\Gamma_1}); \tag{7}$$

$$[v]_0^{D_T} \le C(a, M_1, T)(|f|_0^D + [\psi]_0^{\Omega_1} + [\varphi]_0^{\Gamma_1}) \tag{8}$$

$$(D_T = D \cap \{0 \le t \le T\}).$$

Remark. The right-hand side f in (4), the initial function ψ in (5), and the boundary function φ in (6) are, of course, assumed to be compatible by virtue of equation (4).

Proof. The use of the apparatus of heat potentials (cf. ([7])) makes it possible to write the solution $v(x,t)$ of problem (4)–(6) in explicit form; after this, a sufficiently painstaking investigation of the properties of the heat potentials obtained makes it possible to prove the validity of the estimates (7), (8).

Lemma 3 (cf. ([5,6])) (on the interior $(2+\alpha)$ a priori estimate). Let conditions A_1—A_5 be satisfied with respect to the domain Q and the data of problem (1)–(3). Let problem (1)–(3) have a solution $u(x,t)$, for which $|u|_{2+\alpha}^{\overline{Q}} < +\infty$. Then there exists a constant $C(Q, a, M_1)$ such that, for an interior domain $Q_1 \subset \overline{Q}$, for which $d(Q, Q_1) \geq \lambda > 0$, the estimate

$$|u|^{Q_1}_{2+\alpha} \leq C(Q, a, M_1)(|f|^Q_\alpha + \lambda^{-(2+\alpha)}[u]^Q_0)$$

holds.

Lemma 4 (cf. ([5,6])) (on the boundary $(2+\alpha)$ a priori estimate near Ω). Let the conditions of Lemma 3 be satisfied. Let $G \subseteq Q$ be a domain for which

$$G \cap \Omega = \Omega_G \subset \Omega$$

is an interior subset of the domain Ω ($\overline{\Omega}_G \cap \Gamma = 0$). Then, for any subdomain $G_1 \subset G$ contained in G, there exists a constant $C(Q, \Omega_G, a, M_1)$ such that

$$|u|^{G_1}_{2+\alpha} \leq C(Q, \Omega_G, a, M_1)(|f|^Q_\alpha + |\psi|^{\Omega_G}_{2+\alpha} + [u]^Q_0).$$

§ 3. With the aid of Lemmas 1–4, using the classical method of Schauder (see ([1,2]), as well as ([4])), the following boundary a priori estimates of the solution of problem (1)–(3) are established.

Theorem 1. Let conditions A_1—A_5 be satisfied. Let $Q' \subseteq Q$ be a subdomain of Q (in particular, it may coincide with Q), with

$$\Omega' = \overline{Q'} \cap \Omega, \qquad \Gamma' = \overline{Q'} \cap \Gamma.$$

Let $u(x,t)$ be a solution, bounded in $\overline{Q}$, of problem (1)–(3), in which Ω in (2) and Γ in (3) are replaced respectively by Ω' and Γ'. Finally, let

$$|u|^{\overline{Q}}_{2+\alpha} < +\infty.$$

Then

$$|u|^{Q'}_{2+\alpha} \leq C(Q', Q, a, M_1, M_2, \delta)(|f|^Q_\alpha + |\varphi|^{\Gamma'}_{1+\alpha} + |\psi|^{\Omega'}_{2+\alpha} + [u]^Q_0).$$

Theorem 2. In the case when the domain Q is bounded and the solution $u(x,t)$ of problem (1)–(3) ($|u|^Q_{2+\alpha} < +\infty$) is unique, one has

$$|u|^Q_{2+\alpha} \leq C(|f|^Q_\alpha + |\varphi|^\Gamma_{1+\alpha} + |\psi|^\Omega_{2+\alpha}),$$

where the constant C depends on the equation, but does not depend on the function $u(x,t)$.

§ 4. We now consider the following boundary-value problem for a parabolic system of special form:

$$\sum_{i,j=1}^{n} a_{ij}^{(k)}(x,t)\frac{\partial^2 u_k}{\partial x_i \partial x_j} + \sum_{l=1}^{m}\sum_{i=1}^{n} b_{il}^{(k)}(x,t)\frac{\partial u_l}{\partial x_i} + \sum_{l=1}^{m} c_l^{(k)}(x,t)u_l - \frac{\partial u_k}{\partial t} = f_k(x,t), \tag{9}$$

$$k = 1, 2, \dots, m, \qquad (x,t) \in Q;$$

$$u_k(x,0) = \psi_k(x), \qquad x \in \Omega; \tag{10}$$

$$\frac{\partial u_k(x,t)}{\partial N_k} + \sum_{l=1}^{m} b_l^{(k)}(x,t)u_l(x,t) = \varphi_k(x,t), \qquad (x,t) \in \Gamma, \tag{11}$$

where $\partial/\partial N_k$ is the derivative along the k-conormal to the surface Γ at the point (x,t).

Theorem 3. Suppose that for the system (9)–(11), the domain Q, and the surface Γ, all conditions A_1—A_5 are satisfied. Let a subdomain $Q' \Subset Q$ of the domain Q satisfy the conditions of Theorem 1. Let $u_k(x,t)$ $(k = 1, 2, \dots, m)$ be a solution of problem (9)–(11), bounded in the domain $\overline{Q}$, and such that

$$|u_k|_{2+\alpha}^{\overline{Q}} < +\infty, \qquad k = 1, 2, \dots, m.$$

Then

$$|u_k|_{2+\alpha}^{Q'} < C(Q'; Q, \alpha, M_1, M_2, \delta) \max_k \left(|f_k|_{\alpha}^{Q} + |\varphi_k|_{1+\alpha}^{\Gamma} + |\psi_k|_{2+\alpha}^{\Omega} + [u_k]_0^Q\right).$$

§ 5. With the aid of the established boundary a priori estimate, the existence theorems for a solution of problem (1)–(3) or (9)–(11) are proved in the usual way.

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Note: Figure translations are in progress. See original paper for figures.