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Abstract

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EXPANSION IN EIGENFUNCTIONS OF ORDINARY DIFFERENTIAL OPERATORS WITH IRREGULAR SPLIT BOUNDARY CONDITIONS

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Let us consider the boundary-value problem on the interval $[0,1]$ generated by the differential equation

$$y^{(n)} + p_1(x)y^{(n-2)} + \dots + p_n(x)y + \lambda y = 0 \tag{1}$$

and by the normalized (([1]), p. 51) split boundary conditions:

$$u_i(y) = \sum_{j=0}^{\sigma_i} a_{i\,n-\sigma_i+j}y^{(\sigma_i-j)}(0) = 0 \quad (t = 1,\dots,m),$$

$$u_i(y) = \sum_{j=0}^{\chi_i} b_{i\,n-\chi_i+j}y^{(\chi_i-j)}(1) = 0 \quad (i = m+1,\dots,n). \tag{2}$$

Here

$$n-1 \geq \sigma_1 > \sigma_2 > \dots > \sigma_m \geq 0, \qquad \prod_{i=1}^{m} a_{i\,n-\sigma_i} \neq 0;$$

$$n-1 \geq \chi_{m+1} > \chi_{m+2} > \dots > \chi_n \geq 0, \qquad \prod_{i=m+1}^{n} b_{i\,n-\chi_i} \neq 0;$$

λ is a complex parameter. We assume that $p_i(x)$ $(i = 2,\dots,n)$ are arbitrary complex-valued functions, with

$$\frac{d^{n-i}}{dx^{n-i}}p_i(x) \in C[0,1]$$

$(i = 2,\dots,n)$; $n > m > n-m$. The requirement $n > m > n-m$ is inessential, since the case $0 < m < n-m$ is reduced to ours by the substitution $x = 1-t$. The case $m = n-m$, however, has been well studied (([1]), pp. 52, 67–73), and therefore we shall not consider it. Boundary-value problems of the form (1)–(2)

were studied by M. V. Keldysh ([2]) and L. E. Ward ([3]). Thus, M. V. Keldysh established the completeness of the system of eigenfunctions and associated functions (([1]), pp. 21–24) in $\mathcal{L}_2[0,1]$. In doing so he established completeness, even k-fold completeness, for a more general class of boundary-value problems than (1)–(2). Ward ([3]) investigated the question of expansion in eigenfunctions of the boundary-value problem:

$$y^{(n)} + \lambda y = 0, \quad y(0) = y'(0) = \cdots = y^{(n-2)}(0) = y(1) = 0. \tag{3}$$

He showed that only certain classes of analytic functions can be expanded in uniformly convergent series in eigenfunctions, and gave sufficient conditions for the expansion of arbitrary functions in such series. In the present paper Ward' s results are extended to arbitrary boundary-value problems of the form (1)–(2).

For the boundary-value problem (1)–(2), just as was done in ([1]), pp. 54–61, one can obtain the following theorem on the existence and asymptotics of the eigenvalues.

Theorem 1. *The boundary-value problem (1)–(2) has an infinite set of eigenvalues* λ_k. *For them, for some integer* h*, the following asymptotic formulas hold:*

$$\lambda_{k+h} = \rho_k^n \quad (k = 1, 2, \ldots),$$

where

$$\rho_k = \frac{(2k-1)\pi - m\pi - \dfrac{2\pi}{n}\left(\sum_1^m \sigma_j + \sum_{m+1}^n \chi_j\right)}{2\sin\dfrac{m}{n}\pi} + O\left(\frac{1}{k}\right)$$

for $n - m = 2\mu + 1$,

$$\rho_k = \frac{2k\pi - m\pi - \dfrac{2\pi}{n}\left(\sum_1^m \sigma_j + \sum_{m+1}^n \chi_j\right)}{2\sin\dfrac{m}{n}\pi} e^{i\frac{\pi}{n}} + O\left(\frac{1}{k}\right)$$

for $n - m = 2\mu$. In this case all eigenvalues, beginning with some one, are simple.

For the case $p_2(x) \equiv p_3(x) \equiv \cdots \equiv p_n(x) \equiv 0$, Theorem 1 was established by M. V. Keldysh ([1]), pp. 76-80.

Let $\{\varphi_k(x)\}_{k=1}^{\infty}$ be the sequence of all eigenfunctions and associated functions, numbered in the order of increasing moduli of the eigenvalues. Then, by arguments analogous to the corresponding arguments of ([3]), one obtains the following necessary conditions for uniform convergence of series in the system $\{\varphi_k(x)\}_{k=1}^{\infty}$.

Theorem 2. If the series $\sum_{k=1}^{\infty} a_k\varphi_k(x)$ converges uniformly on some interval $[x_0, x_1]$ $(0 < x_0 < x_1 < 1)$, then:

a) the series

$$\sum_{k=1}^{\infty} a_k \frac{d^q}{dx^q} A^p(\varphi_k(x)) \qquad (p = 0, 1, 2, \dots;\ q = 0, 1, \dots, n-1)$$

converge absolutely and uniformly on every interval $[0, a]$, where $0 < a < x_1$;

b) the sum $f(x)$ of the series $\sum_{k=1}^{\infty} a_k\varphi_k(x)$ is an operator-analytic, in the sense of M. K. Fage ([4]), function in $[0, x_1)$; moreover, if $x \in [0, x_1 - \delta]$ $(0 < \delta < x_1)$, then for some $c > 0$, depending only on δ,

$$\left|\frac{d^q}{dx^q} A^p(f)\right| \le c \left(\frac{1+\delta}{(x - x_1 + \delta)\cos\dfrac{m}{n}\pi}\right)^{pn+q} (pn+q)!$$

$$(p = 0, 1, 2, \dots;\ q = 0, 1, \dots, n-1);$$

c) the functions $f(x), A(f), A^2(f), \dots$ satisfy the boundary conditions at the point 0.

Here $A^p(f) = A^{p-1}(A^1(f))$ for $p > 1$, $A^1(f) = A(f) = f^{(n)}(x) + p_2(x)f^{(n-2)}(x) + \dots + p_n(x)f(x)$, $A^0(f) = f(x)$.

We formulate and outline the proof of the main theorem on expansion in eigenfunctions and associated functions. For boundary-value problem (3), Theorem 3 was established in ([3]).

Theorem 3. Let $f(x)$ from $\mathcal{L}[0,1]$ be an operator-analytic function in $[0, a)$ $(0 < a < 1)$, and let the functions $f(x), A(f), A^2(f), \dots$ satisfy the boundary conditions at the point 0. Suppose, further, that

$$\frac{1}{R} = \lim_{k\to\infty} \sqrt[k]{\frac{|a_k|}{k!}}, \qquad \text{where } a_k = \left(\frac{d^{\,k-n[k/n]}}{dx^{\,k-n[k/n]}} A^{[k/n]}(f)\right)_{x=0}.$$

Then on every interval $[0, b]$ from $[0, a) \cap [0, R)$, $f(x)$ is expanded in a uniformly convergent series in the system $\{\varphi_k(x)\}_{k=1}^{\infty}$.

Proof. Denote by $\{\Gamma_k\}_{k=1}^{\infty}$ an arbitrary sequence of circles in the λ-plane with common center at the point $\lambda = 0$, with radii increasing without bound and such that between any two neighboring circles Γ_k and Γ_{k+1} there lies exactly one eigenvalue. Moreover, suppose that

$$\inf_{\lambda\in\bigcup_1^\infty \Gamma_k,\ \lambda_i\in\Lambda} |\lambda-\lambda_i| > 0,$$

where Λ is the set of all eigenvalues. Further, let $G(x,\xi;\lambda)$ be the Green' s function ((1), pp. 36-37). From the known properties ((1), pp. 38-40) of the function $G(x,\xi;\lambda)$ and Theorem 1 it follows that there exists an integer l such that

$$\frac{1}{2\pi i}\int_{\Gamma_k}\int_0^1 G(x,\xi;\lambda)f(\xi)\,d\xi\,d\lambda = \sum_{j=1}^{k+l} a_j\varphi_j(x),$$

where

$$a_j = \int_0^1 f(\xi)\,\overline{\psi_j(\xi)}\,d\xi$$

and $\{\psi_k(x)\}_{k=1}^\infty$ is the system of functions biorthogonal to the system $\{\varphi_k(x)\}_{k=1}^\infty$. Denote by $\tilde f(x)$ some function from $C^{(n)}[0,1]$, coinciding with $f(x)$ for $x\in[0,b_1]$ ($b<b_1<\min\{R,a\}$) and satisfying (2). Then for $x\in[0,b]$ we have

$$f(x)-\sum_{j=1}^{k+l} a_j\varphi_j(x) = \frac{1}{2\pi i}\int_{\Gamma_k}\int_0^1 GA(f)\,d\xi\,\frac{d\lambda}{\lambda}+\frac{1}{2\pi i}\int_{\Gamma_k}\int_{b_1}^1 G[\tilde f(\xi)-f(\xi)]\,d\xi\,d\lambda. \tag{4}$$

With the aid of the asymptotic formulas for the solution of equation (1) ((1), pp. 45-48), one can show that, for $\lambda\in\bigcup_{k=1}^\infty\Gamma_k$ and some $C>0$ and $\Delta>0$, independent of x,ξ,λ,

$$|G(x,\xi;\lambda)|_{\xi\geq x}\leq C|\lambda|^{-\frac{n-1}{n}}e^{-\Delta|\lambda|(\xi-x)^{1/n}}. \tag{5}$$

On the other hand, using the expansion of the function $A(f)$ in a generalized Taylor series (4), we obtain that, for $\lambda\in\bigcup_{k=1}^\infty\Gamma_k$ and some $C>0$, independent of λ and x,

$$\left|\int_0^x G(x,\xi;\lambda)A(f)\,d\xi\right|\leq C|\lambda|^{-\frac{1}{n}}. \tag{6}$$

From (4)–(6) follows the validity of Theorem 3.

Theorem 3 is sharp in the sense that there exists a boundary-value problem of the form (1)–(2) such that for every $\gamma>0$ one can specify an operator-analytic function $f_\gamma(x)$ on the whole interval $[0,1]$, which satisfies the conditions of Theorem 3, but for which the series

$$\sum_{j=1}^\infty a_j\varphi_j(x),\qquad a_j=\int_0^1 f_\gamma(\xi)\overline{\psi_j(\xi)}\,d\xi,$$

diverges at some points of the interval $(R, R+\gamma)$.

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Note: Figure translations are in progress. See original paper for figures.