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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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ON THE STRUCTURE OF A MAGNETIC TOROIDAL FIELD

THAT DOES NOT POSSESS MAGNETIC SURFACES

In papers ([1,2]), by means of numerical integration it was shown that, under a "resonant" perturbation of a three-turn helical magnetic field ($\mathbf{H} = \nabla\psi$) $\psi_3 = H_0 z + h_3 I_3(3r)\sin 3(\varphi - z)$ by a corrugated field $\psi_0 = h_0 I_0(3r)\sin 3z$ of sufficiently large amplitude h_0, the magnetic surfaces* begin to be visibly destroyed. Namely, if through the image points of a field line (i.e., through the sequence of points of intersection of a field line with the plane $z = 0$) one draws a smooth curve, then this curve is, generally speaking, not closed (Fig. 1, ,).

Fig. 1

In the present work a numerical calculation was carried out for a large number of field lines of the field $\psi_3 + \psi_0$ for $H_0 = 1$, $h_3 = 3$, $h_0 = 0.120;\ 0.125;\ 0.130$**. As a result, a number of qualitative and quantitative regularities of the field with disintegrating magnetic surfaces were established.

1. In the work, graphs were obtained of the radial displacement $\delta = \delta(r)$ of an image point of a field line after a circuit around the center of the petal O'; the initial points were taken on the segment OO', see Fig. 2 (we assumed $\delta(r) = r' - r$, where $M(r)$ is the initial point of the field line, and $M(r')$ is the point at which the smooth curve drawn through $M(r)$ and the subsequent image points intersects the segment OO' after a circuit around O'***).

From Fig. 2 it is seen that δ can take values of either sign. As $r \to 0$ the amplitude and frequency of the oscillations of the function $\delta(r)$ increase; begin-

Fig. 2

Figure 2: Fig. 2

ning with some $r = r^*$ ("the first critical distance") intervals appear on which δ is, as it were, equal to infinity. In fact, $\delta = \infty$ means that the sequence of image points leaves the neighborhood of the point O' and can return to this neighborhood only after a circuit around the point O. The alternation of finite and "infinite" values of δ for $r < r^*$ indicates a very strong entanglement of the field lines.

* As in ([1,2]), we consider one period of the field as a torus and by a magnetic surface understand a closed toroidal surface on which one field line is wound everywhere densely.
** On the method and accuracy of the calculation, see ([1,2]).
*** We note that as $h_0 \to 0$ the radial displacement decreases exponentially; see below, item 3, and also ([3]).

It is also seen from Fig. 2 that, as one approaches the center of the lobe O', $\delta(r)$ tends to zero, while the angle α at which the curve intersects the ray OO' decreases monotonically. As is readily seen, any point $M(r)$ on OO' for which $\delta(r) = 0$ and $|\delta'(r)| < 2$ is stable for $\delta'(r) < 0$ and unstable for $\delta'(r) > 0$. Namely, if one takes a point on OO' sufficiently close to $M(r)$, then after a circuit around O' the representative point will, in the case $\delta'(r) < 0$, approach $M(r)$, and in the case $\delta'(r) > 0$, move away from it.

Fig. 2

The smallest $r = r^{**}$ for which $\delta(r^{**}) = 0$ and $|\delta'(r^{**})| < 2$ we shall call the second critical distance.

2. It has been established that the points $M(r)$ for which $\delta(r) = 0$ correspond to periodic solutions. Namely, starting from such a point, after a circuit around O' we arrive exactly on the x-axis, i.e. again at the same point. This fact was established in the following way.

Graphs of the dependence of the angle φ on r were constructed for various N, where r is the initial coordinate of the point on the x-axis and N is the number of periods traversed by the field line; see Fig. 3. (In this figure the graph of the function $\delta(r)$ is shown by a solid line, and the graphs $\varphi_N(r)$ by dashed lines.) It turned out that the points at which $\delta = 0$ do indeed have $\varphi = 0$ for $N = N_{\text{return}}$,

corresponding to the return to the x-axis. In what follows we shall call the points $M(r)$ for which $\delta = 0$ rational points.

Let us note that in the work only those rational points were considered which lie on the x-axis and which go into themselves after one revolution about the point O'. It has been established that on an arbitrary ray issuing from the point O, generally speaking, there are no rational points. The structure of the set of

Fig. 3

Figure 3: Fig. 3

all rational points in the plane $z = 0$ remains as yet undetermined.

Fig. 3

It has been established (see Fig. 3) that for neighboring rational points the values of N_{return} differ by unity. The calculation shows that the smallest value of N_{return} is 12, and therefore on the segment $O'O$, N_{return} varies from 12 to ∞.* Thus, for any $N \geqslant 12$ on the segment OO' there is at least one point which, after N periods, goes into itself (the period is taken to be $2\pi/3$).

It might seem that a closed magnetic surface must pass through the field line issuing from a rational point. However, this is not so, since such a surface, by its definition, would be "opaque" both for external and for internal field lines, which for small r contradicts Fig. 2 (for r close to r_0, the existence of such surfaces is not excluded). The fact that for small r the rational points do not lie on a magnetic surface was also established by numerical computation (see ([2]), where irrational points were considered; for rational points an analogous picture obtains).

3. On the basis of Figs. 2 and 3 one can trace how, with r (for various h_0), the positive maxima A of the function $\delta(r)$, the distances λ between

* The fact that $N_{\text{return}} \to \infty$ as $r \to 0$ follows directly from the differential equations of the field.

these positive maxima and the number N_{return}. It is indicated that the coordinates of the point O' and λ are, roughly speaking, linear functions of h_0, while A, r^*, and $N_{\text{return}}(r^*)$ change with changing h_0 exponentially. From what has been said it follows that, with a further increase of h_0, the pattern of field lines will change strongly and, beginning from some $(h_0)_{\text{crit}}$, will become chaotic to the highest degree. This $(h_0)_{\text{crit}}$ can be determined by means of one of the conditions: 1) $r^* \sim r_{O'}$; 2) $A(r^*) \sim r_{O'}$; 3) $N_{\text{return}}(r^*) \sim 12$. All these conditions give $h_0 \sim 0.14$. The calculation of individual field lines carried out for $h_0 = 0.14$ confirmed the consideration stated above.

4. The question arises as to what is stable in the unstable magnetic field considered by us. It turned out that in this field there are two families of surfaces possessing the following property: over one period of the field, the surfaces of each family pass into one another.* The existence of such surfaces was established in the following way. On the plane $z = 0$, individual points were taken, and at them two arbitrary directions were prescribed. It turned out that, in passing from the initial point M_0 to the image points M_N (N is the number of periods), these directions always draw closer together, and for large N they practically merge. Thus, on the plane $z = 0$ there arises a field of directions; the integral curves of this field define one

of the sought families of surfaces (the other family is obtained when the sign of z is changed).

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* The existence of such surfaces for certain types of dynamical systems was established in ([4]).

Note: Figure translations are in progress. See original paper for figures.