REPRESENTATION OF THE IMAGINARY PARTS OF AMPLITUDES FOR MANY-PARTICLE INTERMEDIATE STATES

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Abstract

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A. M. BRODSKII

REPRESENTATION OF THE IMAGINARY PARTS OF AMPLITUDES FOR MANY-PARTICLE INTERMEDIATE STATES

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In the paper the unitarity condition is written down for the general case of many-particle intermediate states in a form that makes it possible to extract, under the integral on the right-hand side, as a simple factor the dependence on the transferred momentum (formula (10)). The formula obtained can be used to elucidate the analytic properties of scattering amplitudes with respect to the transferred momentum for the case of many-particle intermediate states. The structure of the indicated formula further makes it possible to represent the imaginary part of the amplitude in the form of the imaginary part of a certain analytic function, whose singularities are determined essentially by the scattering kinematics. No assumptions are used here except the validity of the unitarity condition and the boundedness of the amplitudes in the physical region.

Consider scattering with two particles (I and II) in the initial (i) state and with two particles (III and IV) in the final (f) state, with energy W in the c.m. system and with the cosine of the scattering angle $\cos\varphi_{\mathrm{I\,III}} = z_{\mathrm{I\,III}}$ (the generalization to the case of a larger number of particles in the initial and final states presents no fundamental difficulties in the present case). We shall characterize each scattering problem ($a \to b$) by a set of scalar amplitudes f_λ^{ba} ($\lambda = 1, \dots, \lambda_{ba}$), which are obtained after writing the scattering T-matrix as a sum of products of f_λ^{ba} with invariant matrices in spin, isospin, etc., spaces, according to the kinematics of the problem. The indicated matrices may be chosen: a) in the form of polynomials in the momenta, in most problems no higher than first degree depending on the number of polarization vectors; b) Hermitian; and c) so that the f_λ^{ba} contain no kinematic infinities ([1]). In this case the half-sum of the unitarity conditions for the direct ($i \to f$) and inverse ($f \to i$) processes is written in the c.m. system in the form

$$\frac{1}{2}\,\mathrm{Im}\left(f_\lambda^{fi} + f_\lambda^{if}\right) =$$

$$= \frac{1}{2\pi} \sum \int \frac{(d\bar{\mathbf{p}}_1) \ldots (d\bar{\mathbf{p}}_{N-1})}{(2W_1) \ldots (2W_N)} \, \delta \left(\sum_{i=1}^{N} W_i - W \right) z_{\mathrm{I\,III}}^k \ \mathrm{Re} \left[a_{n\lambda'\lambda''\lambda}^{fi\,(k)} f_{\lambda'}^{fn} f_{\lambda''}^{in*} \right], \quad (1)$$

$$\sum = \sum_n \cdot \sum_{\lambda'} \cdot \sum_{\lambda''} \cdot \sum_k .$$

Here n is the index of the intermediate state; $N = N(n)$ is the number of particles in the state n; W_i is the energy and $\bar{\mathbf{p}}_i$ the three-dimensional momentum of the i-th particle.

The polynomial

$$\sum_{k=0}^{m(n)} z_{\mathrm{I\,III}}^k \, a_{n\lambda'\lambda''\lambda}^{fi\,(k)}$$

in $z_{\mathrm{I\,III}}$, with coefficients satisfying the condition

$$\left[a_{n\lambda'\lambda''\lambda}^{fi\,(k)} \right]^* = a_{n\lambda'\lambda''\lambda}^{if\,(k)},$$

is the coefficient matrix for the transition from the product of the matrices corresponding to $f_{\lambda'}$ and $f_{\lambda''}$, to the complete set of matrices corresponding to f_λ, and is determined by the kinematics of the problem. The amplitudes f_λ^{fi} and f_λ^{if} standing on the left in (1) can be expressed linearly in terms of one another by virtue of invariance with respect to time reversal. Each of the amplitudes $f_{\lambda'}^{fn}$, $f_{\lambda''}^{in*}$ is a function of $3(N+2)-10$ independent variables. As these variables we shall take kinematically and geometrically mutually independent ([2])

sets, which we shall call chain sets. For $f_{\lambda'}^{fn}$ a chain set has, in the c.m. system, the following form:

$$z_{12}, z_{23}, \ldots, z_{i-1,i};\ z_{i,i+1}, \ldots, z_{N-2,N-1}, \quad z_{\mathrm{III}\,1}, z_{\mathrm{III}\,2}, \ldots, z_{\mathrm{III},N-1}; \quad \overline{\mathbf{p}}_1^2, \overline{\mathbf{p}}_2^2, \ldots, \overline{\mathbf{p}}_{N-1}^2. \quad (2)$$

The digits $1, 2, \ldots$ number the particles in the state n. For $f_{\lambda''}^{in}$ the chain set of variables has an analogous form, with III replaced by I. In order to isolate in (1) the dependence on $z_{I\mathrm{III}}$, it is necessary to pass from the integration variables $\overline{\mathbf{p}}_1, \ldots, \overline{\mathbf{p}}_{N-1}$ to new variables:

$$\overline{\mathbf{p}}_1^2, \ldots, \overline{\mathbf{p}}_{N-1}^2; \quad z_{\mathrm{III}\,1}, \ldots, z_{\mathrm{III},N-1}; \quad z_{I1}, \ldots, z_{I,N-1}. \quad (3)$$

In this case the cosines of the angles $z_{12}, z_{23}, \ldots, z_{N-2,N-1}$ remaining among the arguments of the amplitudes $f_{\lambda'}^{in}$ and $f_{\lambda''}^{fn}$ are expressed, by virtue of their geometrical dependence, through (3) by setting equal to zero the Gram determinants $G^{(3)}$, divided by the squares of the vectors entering into them:

$$\overline{G}(\mathrm{I},\mathrm{III},i,k) \equiv \frac{G(\overline{\mathbf{p}}_I,\overline{\mathbf{p}}_{\mathrm{III}},\overline{\mathbf{p}}_i,\overline{\mathbf{p}}_k)}{\overline{\mathbf{p}}_I^2\overline{\mathbf{p}}_{\mathrm{III}}^2\overline{\mathbf{p}}_i^2\overline{\mathbf{p}}_k^2} \equiv \begin{vmatrix} 1 & z_{I\mathrm{III}} & z_{Ii} & z_{Ik} \\ z_{\mathrm{III\,I}} & 1 & z_{\mathrm{III}\,i} & z_{\mathrm{III}\,k} \\ z_{i\mathrm{I}} & z_{i\mathrm{III}} & 1 & z_{ik} \\ z_{k\mathrm{I}} & z_{k\mathrm{III}} & z_{ki} & 1 \end{vmatrix} = 0$$

$$(i,\ k = 1,\dots,N-1). \tag{4}$$

When $z_{i,k+1}$ from (4) is substituted into the arguments of the amplitudes, the cosine of the scattering angle $z_{I\mathrm{III}}$, whose dependence we have to isolate, will enter. * To avoid this difficulty, we shall formally assume that the $N-1$ vectors of the intermediate particles $\overline{\mathbf{p}}_i$ are no longer three-dimensional, and that to each vector, except the first, a fourth independent orthogonal component $\overline{\mathbf{p}}_{i4}$ $(i = 2,\dots,N-1)$ is added. We shall carry out the integration in the unitarity condition over all four components $\overline{\mathbf{p}}_i$, and for compensation we shall introduce into the integral the product of $(N-2)$ δ-functions:

$$\Pi_\delta \equiv \delta(\overline{\mathbf{p}}_{24})\dots\delta(\overline{\mathbf{p}}_{N-1,4}) = \delta\left(\sqrt{\frac{\overline{G}(\mathrm{I},\mathrm{III},1,2)\,\overline{\mathbf{p}}_2^2}{\overline{G}(\mathrm{I},\mathrm{III},1)}}\right)\times$$

$$\times\delta\left(\sqrt{\frac{\overline{G}(\mathrm{I},\mathrm{III},2,3)\,\overline{\mathbf{p}}_3^2}{\overline{G}(\mathrm{I},\mathrm{III},2)}}\right)\dots\delta\left(\sqrt{\frac{\overline{G}(\mathrm{I},\mathrm{III},N-2,N-1)\,\overline{\mathbf{p}}_{N-1}^2}{\overline{G}(\mathrm{I},\mathrm{III},N-2)}}\right). \tag{5}$$

Here we have used a formula of the general theory of Gram determinants [3], which allows one to express the value of the component of a vector in a direction orthogonal to the remaining vectors composing the given determinant. After the described introduction of the additional dimensions, we obtain the right to add to the variables (3) additional $N-2$ independent integration variables: $z_{12}, z_{23},\dots,z_{N-2,N-1}$, and, correspondingly, not to introduce $z_{I\mathrm{III}}$ into the arguments of the amplitudes. As a result, after passing to the new integration variables, we can write (1) in the form

$$(1) = \frac{1}{2\pi}\sum\int\left\{\frac{d\overline{\mathbf{p}}_1^2\dots d\overline{\mathbf{p}}_{N-1}^2\,dz_{I1}\dots dz_{I,N-1}\,dz_{\mathrm{III}\,1}\dots dz_{\mathrm{III},N-1}\,dz_{12}dz_{23}\dots dz_{N-2,N-1}}{\sqrt{\overline{G}(\mathrm{I},\mathrm{III},1)}\sqrt{\overline{G}(\mathrm{I},\mathrm{III},1,2)\overline{G}(\mathrm{I},\mathrm{III},2,3)\dots\overline{G}(\mathrm{I},\mathrm{III},N-2,N-1)}}\times\right.$$

$$\left.\times\sqrt{\overline{\mathbf{p}}_1^2\dots\overline{\mathbf{p}}_{N-1}^2}\,\Pi_\delta\,z_{I\mathrm{III}}^k A_{n\lambda'\lambda}^{(k)}\right\} =$$

$$= \frac{1}{2\pi}\sum\int\left\{\sqrt{\overline{\mathbf{p}}_1^2\dots\overline{\mathbf{p}}_{N-1}^2}\,\frac{\sqrt{\overline{G}(\mathrm{I},\mathrm{III},1)\dots\overline{G}(\mathrm{I},\mathrm{III},N-2)}}{\sqrt{\overline{G}(\mathrm{I},\mathrm{III},1)}}\,\delta(\overline{G}(\mathrm{I},\mathrm{III},1,2))\times\right.$$

$$\left.\times\delta(\overline{G}(\mathrm{I},\mathrm{III},2,3))\dots\delta(\overline{G}(\mathrm{I},\mathrm{III},N-2,N-1))z^k_{\mathrm{IIII}}A^{(k)}_{n\lambda'\lambda}\,d\Omega\right\}. \tag{6}$$

* The two-valuedness of the solution of (4) turns out to be immaterial owing to the invariance of the unitarity

where

$$A^{(k)}_{n\lambda'\lambda''\lambda} = \frac{\delta\left(\sum_{i=1}^N W_i - W\right)}{(2W_1)\dots(2W_N)}\,\mathrm{Re}\left[a^{i(k)}_{n\lambda'\lambda}f^n_{\lambda'}f^{in*}_{\lambda''}\right]$$

and $d\Omega$ denotes the product of the integration differentials. The integration over $\overline{\mathbf{p}}_i^2$ in (6) is carried out from 0 to ∞, and over z_{ik}, generally speaking, over the region defined by the conditions

$$\overline{G}(I,III,i) > 0, \qquad i = 1,\dots,N-1. \tag{7}$$

It turns out, however, that it is sufficient to require that condition (7) be fulfilled only for one i, for example for $i = 1$, since if the remaining limits of integration are extended along the real axis, the fulfillment of (7) for $i = 2,\dots,N-1$ will be ensured by the δ-functions of the Gram determinants entering (6). Indeed, the indicated δ-functions can be rewritten in the form

$$\delta(\overline{G}(I,III,i,i{+}1)) = \delta\left(\left\{z_{i,i+1}\overline{G}(I,III) - \begin{vmatrix} 1 & z_{I\,III} & z_{I,i+1} \\ z_{III\,I} & 1 & z_{III,i+1} \\ z_{Ii} & z_{IIIi} & 0 \end{vmatrix}\right\}^2 - \overline{G}(I,III,i)\,\overline{G}(I,III,i+1)\right). \tag{8}$$

Passing successively along the chain of δ-functions in (6), we see that their arguments can simultaneously vanish for real cosines only when all the conditions (7) are fulfilled, or, conversely, when all the corresponding Gram determinants are less than zero*. Thus, we can rewrite (6) in the form

$$
\begin{aligned}
(1) = \frac{1}{2\pi} \sum \Big\{ \int & \sqrt{\overline{\mathbf{p}}_1^{\,2} \ldots \overline{\mathbf{p}}_{N-1}^{\,2}}\; \theta(\overline{G}(I, III, 1)) \\
& \times \frac{\sqrt{\overline{G}(I, III, 1) \ldots \overline{G}(I, III, N-2)}}{\sqrt{\overline{G}(I, III, 1)}}\, \delta(\overline{G}(I, III, 1, 2)) \ldots \\
& \ldots \delta(\overline{G}(I, III, N-1, N-2))\, z_{I\,III}^{k} A^{(k)}_{n\lambda'\lambda''\lambda}\, d\Omega \Big\}, \qquad (9)\\
\theta(x) = & \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases}
\end{aligned}
$$

where the limits over all cosines may be taken as equal to ± 1. They could have been extended still further, by including larger intervals of the real axis.

Let us now single out in (9) separately the terms corresponding to one- and two-particle states, and denote, in the multiparticle terms, the integrand expression except for the last δ-function by $B^{(k)}_{n\lambda'\lambda''\lambda}(z_{I\,III})$.

$$
\begin{aligned}
(1) = & \text{ one- and two-particle terms} \\
& + \sum_{N>2} \int B^{(k)}_{n\lambda'\lambda''\lambda}(z_{I\,III})\, \delta(\overline{G}(I, III, N-1, N-2))\, d\Omega \\
= & \text{ one- and two-particle terms} \\
& + \sum_{N>2} \int \Big\{ B^{(k)}_{n\lambda'\lambda''\lambda}(z^{(+)}_{I\,III}) \delta(z_{I\,III} - z^{(+)}_{I\,III}) \\
& \quad + B^{(k)}_{n\lambda'\lambda''\lambda}(z^{(-)}_{I\,III}) \delta(z_{I\,III} - z^{(-)}_{I\,III}) \Big\} \frac{d\Omega}{2\sqrt{\overline{G}(I, N-2, N-1)\, \overline{G}(III, N-2, N-1)}}.
\end{aligned}
\qquad (10)
$$

Here $z^{(\pm)}$ are the roots of the equation $\overline{G}(I, III, N-2, N-1) = 0$, taking values from -1 to $+1$. With the chosen limits of integration over z_{ki}, the quantities $B^{(k)}_{n\lambda'\lambda''\lambda}(z^{(\pm)}_{I\,III})$ are real, and we can rewrite the last

In this case only isolated points, which give no contribution to the integral, are excluded, where the Grammi

terms (10) in the following way:

$$
(1) = \ldots - \frac{1}{\pi} \operatorname{Im} \sum_{N>2} \int \left[- \frac{B^{(k)}_{\pi\lambda'\lambda''}(z^{(+)}_{1\mathrm{III}})}{z_{1\mathrm{III}} + i\varepsilon - z^{(+)}_{1\mathrm{III}}} + \frac{B^{(k)}_{\pi\lambda'\lambda''}(z^{(-)}_{1\mathrm{III}})}{z_{1\mathrm{III}} + i\varepsilon - z^{(-)}_{1\mathrm{III}}} \right] \frac{d\Omega}{2\sqrt{\bar{G}(\mathrm{I}, N-2, N-1)\, \bar{G}(\mathrm{III}, N-2, N-1)}}. \qquad (11)
$$

In the form of the imaginary part of a certain function with a simple dependence on $z_{1\mathrm{III}}$, the one- and two-particle terms in the unitarity condition may also be written. The former are represented in an obvious way as the imaginary parts of pole terms of the theory of dispersion relations, while the latter, if the usual notation is somewhat transformed, may be represented in the form

$$-\frac{1}{2\pi}\,\mathrm{Im}\sum_{N=2}\frac{\bar{p}_{n_2}}{W}\int_{-1}^{+1}dz_{\mathrm{I}1}\int_{-1}^{+1}dz_{\mathrm{III}1}\times$$

$$\times\frac{z_{1\mathrm{III}}^{k}\,\mathrm{Re}\Big[a^{(k)}_{n_2\lambda'\lambda''\lambda}f^{fn_2}_{\lambda}(z_{\mathrm{III}1})f^{in_2*}_{\lambda''}(z_{\mathrm{I}1})\Big]}{\sqrt{[z_{1\mathrm{III}}+i\varepsilon-\cos(\varphi_{\mathrm{I}1}+\varphi_{1\mathrm{III}})][z_{1\mathrm{III}}+i\varepsilon-\cos(\varphi_{\mathrm{I}1}-\varphi_{1\mathrm{III}})]}}. \tag{11'}$$

The addition to (11) and (11′) of an infinitely small imaginary term leads to the corresponding formulas being correct, as can be checked, also for the case of forward and backward scattering. It is interesting to note that in the present case the appearance of infinitely small imaginary components in the integrand is simply a formal consequence of the special behavior of the unitarity integral at $z_{1\mathrm{III}}^2 = 1$. If in (11) and (11′), instead of $z_{1\mathrm{III}} + i\varepsilon$, one substitutes an arbitrary complex quantity ζ, then under the sign of the imaginary part we obtain a function analytic for $\mathrm{Im}\,\zeta \neq 0$, whose limiting value from above for the segment ± 1 gives the physical unitarity condition. The indicated analytic function has only the following singularities: a cut from -1 to $+1$, corresponding to branch points; poles arising from one-particle terms; and possible kinematic poles at infinity (because of $a^{(k)}_{n_2\lambda'\lambda''\lambda}z^k$), determined by two-particle terms. Thus we obtain the formal possibility of continuing the imaginary part of the scattering amplitude into the unphysical region of transferred momentum in the physical energy interval in the following way. First the analytic function entering (11), (11′) is analytically continued. Then the imaginary part is taken from the continuation obtained.* The continuation obtained in this way may be substituted into the dispersion relations. In this case there is excluded the requirement, motivated only by reference to perturbation theory, of separate analyticity of the imaginary and real parts of the scattering amplitude.

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Institute of Petrochemical Synthesis
Academy of Sciences of the USSR

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* The indicated analytic function can evidently be defined not only as was done. The ambiguity of the definition reduces to the fact that we may continue the integration limits over the intermediate z_{ik} from the segment ± 1 in such a way as to capture the singularities of the amplitudes entering $A^{(k)}_{n\lambda'\lambda''\lambda}$. Physically this ambiguity may be interpreted as a consequence of the fact that the real part of the amplitude, unlike the imaginary part, depends more essentially on the detailed character of the interaction, which was not included in the preceding consideration.

Note: Figure translations are in progress. See original paper for figures.