



---

Soviet-era science, translated into English

# Mathematics

1963

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196301.07675>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**Mathematics**

**Ivo Babuška**

**Optimal Quadrature Formulas**

*(Presented by Academician S. L. Sobolev on 11 X 1962)*

### 1. Introduction.

In recent years a number of works have been published concerning optimal formulas. We mention papers <sup>(1-3)</sup>, where the problem of optimizing quadrature formulas in various classes of functions was considered.

The present paper is devoted to the optimization of one-dimensional quadrature formulas for an equidistant partition in the classes  $W_2^{(p)}(T)$ , i.e., the Sobolev spaces  $W_2^{(p)}$  on the interval  $\langle 0, T \rangle$ . The method and the general conclusions can without difficulty be transferred to the general case in  $W_2^{(p)}$  for  $n$ -dimensional quadratures.

Let  $q + 1$  rows of numbers  $\{ {}_l a_k \}$  be given,  $k = 0, 1, \dots, n$ ;  $l = 0, 1, \dots, q \leq p - 1$ . Let  $\{ {}_l a_k \} \in {}^q K_n^p(T)$  if

$$\sum_{k=0}^n \sum_{l=0}^q \left( \frac{T}{n} \right)^{l+1} {}_l a_k f^{[l]} \left( \frac{kT}{n} \right) = \int_0^T f(x) dx$$

for every polynomial of degree not exceeding  $(p - 1)$ .

Let  $\{ {}_l a_k \} \in {}^q K_n^p(T)$ . Then

$$Q_T^{(q)}(\{ {}_l a_k \}) = \sup_{\|f\|_{W_2^{(p)}(T)}=1} \left| \sum_{k=0}^n \sum_{l=0}^q \left( \frac{T}{n} \right)^{l+1} {}_l a_k f^{[l]} \left( \frac{kT}{n} \right) - \int_0^T f(x) dx \right|$$

and  $Q_T^{(q)} < \infty$ .

$\{ {}_l c_k \}$  will be  $(p, q, n, T)$ -optimal if:

- 1)  $\{ {}_l c_k \} \in {}^q K_n^p(T)$ ;
- 2) if  $\{ {}_l a_k \} \in {}^q K_n^p(T)$ ,  $\{ {}_l a_k \} \neq \{ {}_l c_k \}$ , then  $Q_T^{(q)}(\{ {}_l a_k \}) > Q_T^{(q)}(\{ {}_l c_k \})$ .

In paper <sup>(1)</sup> Sard showed that for  $q = 0$ ,  $T = n$  the optimal coefficients  $\{c_k\}$  are determined uniquely, and in paper <sup>(2)</sup> these coefficients were computed for certain values of  $p, n$ . In paper <sup>(3)</sup> S. L. Sobolev gives the differential equation which must be satisfied by the function giving the maximum error of the quadrature for the given coefficients.

## 2. On optimal coefficients.

**Lemma 1.** Let  $\{a_k\} \in {}^q K_n^p(T)$ . Then, up to a polynomial of degree not exceeding  $(p - 1)$ , there exists one and only one function  $u(x) \in W_2^{(p)}$  such that

$$\int_0^T u^{[p]}(x) \varphi^{[p]}(x) dx = (-1)^p \left[ \int_0^T \varphi(x) dx - \sum_{k=0}^n \sum_{l=0}^q \left(\frac{T}{n}\right)^{l+1} a_k \varphi^{[l]} \left(\frac{kT}{n}\right) \right]$$

for every  $\varphi \in W_2^{(p)}(T)$ , and

$$Q_T^{(q)}(\{a_k\}) = \left( \int_0^T (u^{[p]}(x))^2 dx \right)^{1/2}.$$

**Remark 1.** On the intervals  $\left(\frac{kT}{n}, \frac{(k+1)T}{n}\right)$  the function  $u(x)$  is a solution of the differential equation  $d^{2p}u/dx^{2p} = 1$ ; at the points  $kT/n$ ,  $k = 1, \dots, n - 1$ , the derivatives up to order  $2p - q - 2$  inclusive are continuous, and at the points  $0, T$  one has  $d^r v/dx^r = 0$  for  $p \leq r \leq 2p - q - 2$  and  $0 \leq r \leq q$  (cf. also (3)).

**Lemma 2.** Let  $(n + 1)(q + 1) \geq p$ . We have  $f \in W_2^{(p)}(n, q, T)$ , if:

- 1)  $f \in W_2^{(p)}(T)$ ;
- 2)  $f^{[l]} \left(\frac{kT}{n}\right) = 0$  for  $k = 0, \dots, n$ ;  $l = 0, 1, \dots, q$ ;  $0 \leq q \leq p - 1$ .

Then there exists a unique function  $v_q(x) \in W_2^{(p)}(n, q, T)$  such that

$$\int_0^T v_q^{[p]}(x) \psi^{[p]}(x) dx = (-1)^p \int_0^T \psi(x) dx$$

for every  $\psi \in W_2^{(p)}(n, q, T)$ .

**Lemma 3.** The function  $v_q(x)$  is piecewise smooth. If we denote

$$(-1)^l b_k = \frac{d^{2p-l-1} v_q}{dx^{2p-l-1}} \left(\frac{kT}{n} + 0\right) - \frac{d^{2p-l-1} v_q}{dx^{2p-l-1}} \left(\frac{kT}{n} - 0\right) \quad (v_q \equiv 0 \text{ for } x < 0, x > T),$$

then  $\{b_k\} \in {}^q K_n^p$ . Moreover, the function  $v_q(x)$  is equal to the function  $u(x)$  from Lemma 1 for the coefficients  $\{b_k\}$ .

**Theorem 1.** Let  $\{b_k\}$  be the coefficients from Lemma 3. Then  $\{b_k\} = \{c_k\}$ , i.e.  $\{b_k\}$  are  $(p, q, n, T)$ -optimal coefficients.

**Proof.** Let  $\{a_k\} \in {}^q K_n^p$ , and let  $g(x)$  be the function from Lemma 1 corresponding to the coefficients  $\{a_k\}$ . According to Lemmas 1 and 3, it is necessary to prove that

$$\int_0^T (g^{[p]}(x))^2 dx > \int_0^T (v_q^{[p]}(x))^2 dx, \quad \text{if } \{a_k\} \neq \{b_k\}.$$

However

$$\int_0^T v_q^{[p]}(x)(v_q^{[p]}(x) - g^{[p]}(x)) dx = 0,$$

since the functions  $v_q(x)$  and  $g(x)$  have the properties indicated in Remark 1 (cf. Lemma 3). Thus,

$$\int_0^T (v_q^{[p]}(x))^2 dx = \int_0^T v_q^{[p]}(x)g^{[p]}(x) dx. \quad (1)$$

But, according to Schwarz' s inequality,

$$\int_0^T v_q^{[p]}(x)g^{[p]}(x) dx \leq \left( \int_0^T (v_q^{[p]}(x))^2 dx \right)^{1/2} \left( \int_0^T (g^{[p]}(x))^2 dx \right)^{1/2}, \quad (2)$$

with equality only in the case when  $v_q^{[p]}(x) = g^{[p]}(x)$ . From (1) and (2) it follows that

$$\int_0^T (v_q^{[p]}(x))^2 \leq \int_0^T (g^{[p]}(x))^2 dx,$$

with equality only for  $v_q^{[p]}(x) = g^{[p]}(x)$ .

Our assertion follows directly from Lemma 1.

### 3. On the asymptotic behavior of the $(p, q, n, T)$ -optimal coefficients.

**Lemma 4.** Denote by  ${}^r v_q(x)$  the function from Lemma 2 for  $n = T = r$ . Then there exists a function  ${}^0 v_q(x)$  such that  ${}^r v_q(x) \rightarrow {}^0 v_q(x)$  as  $r \rightarrow \infty$  in  $W_2^{[p]}(T')$  for any  $T' < \infty$ , and, consequently,  ${}^r v_q(x) \rightarrow {}^0 v_q(x)$  for every  $x$ , with all its derivatives up to order  $2p - 1$  (at the points  $x = 0, 1, \dots, r$  this refers to left and right derivatives).

The proof is based on ideas similar to those used in (4).

**Lemma 5.** There exist constants  $q \uparrow_s$ ,  $s = 0, \dots, q$ , such that

$$q \uparrow_s = \lim_{r \rightarrow \infty} \frac{d^{2p-1-s} {}^0 v_q}{dx^{2p-1-s}}(r+0) - \frac{d^{2p-1-s} {}^0 v_q}{dx^{2p-1-s}}(r-0).$$

**Theorem 2.** Let  $\{ {}^1 c_k \}$  be the  $(p, q, n, T)$ -optimal coefficients; then

$$\lim_{n \rightarrow \infty} {}^1 c_{[n\alpha]} = q \uparrow_1, \quad 0 < \alpha < 1$$

( $[n\alpha]$  is the integer part of  $n\alpha$ ).

The theorem follows easily from Lemma 5 and Theorem 1 by transforming the interval  $\langle 0, T \rangle$  into the interval  $\langle 0, 1 \rangle$ .

Strictly speaking, Theorem 2 is a statement about the boundary effect for the  $(p, q, n, T)$ -optimal coefficients. The boundary effect can be determined by computing  ${}^0 v_q(x)$ . In the one-dimensional case the coefficients are solutions of certain finite-difference equations.

**Table 1**

	$p = 1$	$p = 2$	$p = 3$
0	0,50000	0,39433757	0,35603685
1	1,00000	1,13397460	1,23176522
2	1,00000	0,96410162	0,87327997
3	1,00000	1,00961894	1,05572300
4	1,00000	0,99742261	0,97595703
5	1,00000	1,00069061	1,01033569
6	1,00000	0,99981495	0,99554153
7	1,00000	1,00004958	1,00191971
8	1,00000	0,99998671	0,99917341
9	1,00000	1,00000356	1,00035590
10	1,00000	0,99999905	0,99984675

In the case  $q = 0$ ,  $p = 1, 2, 3$ , we obtain for

$${}^p\sigma_k = \lim_{r \rightarrow \infty} {}^0c_k(r),$$

where  $\{{}^0c_k(r)\}$  is the  $(p, 0, r, T)$ -optimal coefficient, the numerical values given in Table 1.

As  $p$  increases, the length of the boundary effect increases. Using the usual methods of the calculus of variations (direct methods) to estimate errors, one can obtain an error estimate (i.e. the increase of the quantity  $Q_T^{[p]}$ ) in passing from the infinite boundary effect to a boundary effect of finite length.

Mathematical Institute  
of the Czechoslovak Academy of Sciences

Received  
9 X 1962

### References Cited

1. A. Sard, *Am. J. Math.*, **71**, 80 (1949).
2. F. Meyers, A. Sard, *J. Math. Phys.*, **29**, 118 (1950).
3. . . . , DAN, **137**, No. 3, 527 (1961).
4. I. Babuška, *Czechoslov. Math. J.*, **11**, 76, 165 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*