MATHEMATICS

Yu. M. GORCHAKOV

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

Yu. M. GORCHAKOV

ON INFINITE FROBENIUS GROUPS

(Presented by Academician A. I. Mal' cev on 27 IV 1963)

The following theorem of G. Frobenius is known (see ([1])):

Let $\mathfrak{H}$ be such a proper subgroup of a finite group $\mathfrak{G}$ that from the relation $X^{-1}\mathfrak{H}X \cap \mathfrak{H} \neq 1$ it follows that $X \in \mathfrak{H}$. Then the set of elements of the group $\mathfrak{G}$ not belonging to subgroups conjugate with $\mathfrak{H}$, together with the identity, forms a normal divisor complementing the subgroup $\mathfrak{H}$ in $\mathfrak{G}$.

A subgroup of this kind in an arbitrary group $\mathfrak{G}$ (finite or infinite) will be called **isolated** in $\mathfrak{G}$. A group $\mathfrak{G}$ having an isolated subgroup is called a **Frobenius group**.

In the case of an infinite group $\mathfrak{G}$, the Frobenius theorem may turn out to be false for all or for some isolated subgroups of the group $\mathfrak{G}$ (the author has examples of such groups—one of them is given below).

It is known ([2]) that the Frobenius theorem is valid for all isolated subgroups of locally finite groups; it is also known that it is valid for algebraic (for the definition of an algebraic group see ([3]), p. 109) isolated subgroups of algebraic groups (see ([4])).

In the present note the following sufficient conditions are proposed under which all isolated subgroups are complemented in an infinite group.

Theorem 1. *Let $\mathfrak{H}$ be an arbitrary subgroup isolated in $\mathfrak{G}$ and $\pi = \pi(\mathfrak{H})$ the set of prime divisors of the orders of its elements. Then, for the Frobenius theorem to hold for all isolated subgroups of the group $\mathfrak{G}$, it is sufficient that it contain such a π-complete locally nilpotent normal divisor $\mathfrak{N}$, the factor group $\mathfrak{G}/\mathfrak{N}$ by which is locally finite. Under this condition any two isolated subgroups of the group $\mathfrak{G}$ are conjugate in $\mathfrak{G}$.*

A group $\mathfrak{L}$ is called **π-complete** if the equation $X = Y^n$ is solvable in it for every $X \in \mathfrak{L}$ and every integer n, all prime divisors of which belong to the set π.

Corollary (Kegel). *If $\mathfrak{H}$ is an isolated subgroup of a locally finite group $\mathfrak{G}$, then the Frobenius theorem is valid for $\mathfrak{H}$.*

The following example shows that in the theorem under consideration the condition of π-completeness of the normal divisor $\mathfrak{N}$ is essential.

Example 1. Let $\mathfrak{G}$ be the subgroup of the group $GL(2, K)$, where K is the field of complex numbers, generated by the elements

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & i \end{pmatrix}.$$

The group $\mathfrak{G}$ has isolated 2-subgroups $\{A\}$ and $\{B\}$, but has no 2-complete locally nilpotent normal divisors defining locally finite factor groups of the group $\mathfrak{G}$; it is not difficult to verify that the Frobenius theorem is false for the subgroup $\{A\}$, and true for $\{B\}$.

The following theorem reduces the study of the structure of isolated subgroups of a mixed group to the study of the structure of isolated subgroups of locally finite (and even finite) groups.

Theorem 2. *Let $\mathfrak{H}$ be an isolated subgroup of a group $\mathfrak{G}$ which is an extension of a locally nilpotent group by means of a locally finite one, and let $\pi(\mathfrak{H})$ be the set of prime divisors of the orders of its elements.*

Then every finite subgroup of $\mathfrak{H}$ is isomorphic to an isolated subgroup of some finite group; if $\pi(\mathfrak{H})$ does not contain all prime numbers, then the group $\mathfrak{H}$ is isomorphic to an isolated subgroup of some locally finite group.

It follows from this, in particular, that the Sylow p-subgroups of the group $\mathfrak{H}$ are either locally cyclic groups or (for $p = 2$) generalized quaternion groups.

From this theorem, by virtue of the results of D. Hertzig' s paper ([4]), we obtain the following

Corollary. *An algebraic Frobenius group has a unique minimal Frobenius splitting (for the definition see ([5])), consisting of algebraic subgroups.*

An **algebraic Frobenius group** is an algebraic group containing a proper algebraic isolated subgroup.

The following example shows that $\pi(\mathfrak{H})$ may contain all prime numbers.

Example 2. Let $p_1 = 2, p_2 = 3, \dots, p_n, \dots$ be all prime numbers arranged in increasing order. And let $q_1, q_2, \dots, q_n, \dots$ be such distinct prime numbers that $q_n - 1$ is divisible by the product $p_1 p_2 \cdots p_n$. As is known, the holomorph of a cyclic group of order q_n contains a group $\mathfrak{G}_n$ of order $q_n p_1 p_2 \cdots p_n$. The latter can be represented in the form of the product

$$\mathfrak{G}_n = \{A_n\}(\{H_{1n}\} \times \{H_{2n}\} \times \cdots \times \{H_{nn}\}),$$

where $A_n^{q_n} = H_{1n}^{p_1} = \cdots = H_{nn}^{p_n} = 1$.

Let

$$\widetilde{\mathfrak{G}} = \widetilde{\prod_n} \mathfrak{G}_n$$

be the complete direct product of the groups $\mathfrak{G}_n$, let $\mathfrak{A}$ be the complete direct product of the groups $\{A_n\}$ $(n = 1, 2, ...)$, and let $\mathfrak{A}$ be the direct product of the latter. In the group $\widetilde{\mathfrak{G}}$ take the elements

$$H_i^* = \prod_n H_{in} \quad (i = 1, 2, ...)$$

(in the case when $i > n$, we assume that $H_{in} = 1$). Denote by $\mathfrak{H}$ the group generated by the elements H_i^* $(i = 1, 2, ...)$. Put $\mathfrak{G} = \mathfrak{H}\mathfrak{A}$. Then in the factor group $\mathfrak{G}/\mathfrak{A}$ the subgroup $\mathfrak{H}\mathfrak{A}/\mathfrak{A}$ is isolated. Clearly, $\pi(\mathfrak{H}\mathfrak{A}/\mathfrak{A})$ contains all prime numbers.

In proving Theorems 1 and 2, one has to establish whether the image of some isolated subgroup $\mathfrak{H}$ of a group $\mathfrak{G}$ is isolated in the homomorphic image of the latter. For these purposes the following simple lemma serves.

Lemma. *Let $\mathfrak{G}$ be the semidirect product of a normal divisor $\mathfrak{A}$ and a finite abelian group $\mathfrak{H}$ coinciding with its normalizer $N(\mathfrak{H})$ in $\mathfrak{G}$, and let $\mathfrak{Z}$ be such an invariant subgroup of the center of the group $\mathfrak{A}$ that the order of each element of the factor group $\mathfrak{A}/\mathfrak{Z}$ is relatively prime to the order of the group $\mathfrak{H}$ (we assume that infinite order is relatively prime to any natural number).*

Then $N(\mathfrak{H}\mathfrak{Z}) = \mathfrak{H}\mathfrak{Z}$.

From the lemma it is not difficult to obtain the following propositions, used in the proof of Theorems 1 and 2.

1) Let $\mathfrak{G} = \mathfrak{H}\mathfrak{A}$, $\mathfrak{H} \cap \mathfrak{A} = 1$, where $\mathfrak{H}$ is a locally finite subgroup isolated in $\mathfrak{G}$, and $\mathfrak{A}$ is a locally nilpotent normal divisor, and let $\mathfrak{N}$ be such a normal divisor of the group $\mathfrak{G}$ contained in $\mathfrak{A}$ that the factor group $\mathfrak{A}/\mathfrak{N}$ contains no elements whose orders are divisible

to prime divisors of the orders of elements of the group $\mathfrak{H}$ (we assume that the infinite order is not divisible by any integer).

Then the group $\mathfrak{H}\mathfrak{A}/\mathfrak{N}$ is isolated in $\mathfrak{G}/\mathfrak{N}$.

2) Let $\mathfrak{G} = \mathfrak{H}\mathfrak{A}$, $\mathfrak{H} \cap \mathfrak{A} = 1$, and let $\mathfrak{H}$ be a locally finite isolated subgroup in $\mathfrak{G}$, while $\mathfrak{A}$ is an abelian π-complete normal divisor $(\pi = \pi(\mathfrak{H}))$.

Then $\mathfrak{A}$ consists of elements of the form $H^{-1}A^{-1}HA$, where $A \in \mathfrak{A}$ and H is an arbitrary element of $\mathfrak{H}$ distinct from 1.

Sverdlovsk Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
3 IV 1963

REFERENCES

1. G. Frobenius, Sitzungsber. d. Königl. preuss. Akad. d. Wissensch. Berlin, phys.-math. Classe, 1216 (1901).

2. O. Kegel, Arch. der Math., **13**, No. 1–3, 10 (1961).

3. K. Shmel' lev, Theory of Groups, 2, Izd., 1958.

4. D. Hertzig, Am. J. Math., **83**, 421 (1961).

5. R. Baer, Math. Zs., **75**, 333 (1961).

Note: Figure translations are in progress. See original paper for figures.