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Abstract

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MATHEMATICS

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### ESTIMATES FOR THE DIAMETER AND WIDTH OF CONVEX SURFACES OF BOUNDED GAUSSIAN CURVATURE

(Presented by Academician S. L. Sobolev, June 15, 1963)

1. The main result of this work is the following

**Theorem 1.** *Let the Gaussian curvature of a convex surface  $S$  in  $(n + 1)$ -dimensional Euclidean space  $R^{n+1}$  be contained between  $1 - \varepsilon$  and  $1 + \varepsilon$ . Then the diameter of  $S$  is not greater than  $2 + C\varepsilon$ , and its width is not less than  $2 - C\varepsilon$ , where  $C$  is a constant depending only on  $n$ ; moreover, the order of these estimates cannot be improved.*

2. We shall first study surfaces of revolution. Refer  $R^{n+1}$  to Cartesian coordinates  $x^1 = x, x^2 = y, x^3, \dots, x^{n+1}$ . Let  $y = y(x) \geq 0$  be a curve; the surface of revolution in  $R^{n+1}$  with directrix  $y(x)$  is called the surface of the form  $r = y(x)$ , where

$$r = \left[ \sum_{i=2}^{n+1} (x^i)^2 \right]^{1/2};$$

the principal curvatures of such a surface at the point with abscissa  $x$  are

$$k_1(x) = -\frac{y''(x)}{[1 + y'^2(x)]^{3/2}}, \quad k_2(x) = \dots = k_n(x) = \frac{1}{y(x)[1 + y'^2(x)]^{1/2}};$$

therefore the Gaussian curvature is

$$k(x) = -\frac{y''(x)}{[y(x)]^{n-1}[1 + y'^2(x)]^{n/2+1}}.$$

In what follows we shall consider only such functions  $y(x)$  for which, in the domain of their definition, the inequalities  $y(x) \geq 0, y'(x) \leq 0, y''(x) < 0$  hold. From these conditions it follows, in particular, that the function  $y(x)$

has an inverse  $x = x(y)$ , and that  $k(x) > 0$ . Consider two curves  $y = y_i(x)$  ( $i = 1, 2$ ); denote by  $x_i(y)$  the inverse function for  $y_i(x)$ ; and by  $k_i(x)$  the Gaussian curvature of the surface formed by rotating the curve  $y = y_i(x)$ ; put  $k_i(y) = k_i(x(y))$ .

**Lemma 1.** *Let, for the curves  $x = x_i(y)$  ( $i = 1, 2$ ), defined on the segment  $\alpha \leq y \leq \beta$ , the following conditions be satisfied:*

- 1)  $x_1(\alpha) \geq x_2(\alpha)$  (respectively  $x_1(\beta) \leq x_2(\beta)$ );
- 2)  $k_1(y) \leq k_0 \leq k_2(y)$ , where  $k_0$  is a constant ( $k_1(y) \geq k_0 \geq k_2(y)$ );
- 3)  $|x'_2(\alpha)| \leq |x'_2(\beta)|$  ( $|y'_1(\beta)| \geq |y'_2(\beta)|$ ).

Then for every  $y \in [\alpha, \beta]$  the following assertions are valid:

- 1)  $|y'_1(x_1(y))| \geq |y'_2(x_2(y))|$  ( $|x'_1(y)| \leq |x'_2(y)|$ );
- 2)  $x_2(y) \leq x_1(y)$ , ( $x_2(y) \geq x_1(y)$ ).

**Proof** of Lemma 1 is based on the relation

$$\int_{\xi}^{\eta} k(x)y^{n-1}(x)y'(x) dx = \frac{1}{[1+y'^2(\eta)]^{n/2}} - \frac{1}{[1+y'^2(\xi)]^{n/2}}.$$

If  $k(x) \geq k_0$ , then the left-hand side is  $\leq k_0 [y^n(\eta) - y^n(\xi)]$ , and an analogous estimate is valid for  $k(x) \leq k_0$ , after which the matter reduces to first-order differential inequalities.

From Lemma 1 it follows that

**Lemma 2.** *Let the curve  $x = x(y)$  be defined on  $[0, \beta]$  and satisfy the conditions  $k(x) \leq b^n$ ,  $x'(0) = 0$ ,  $x(0) \geq 1/b$ . Construct in the plane  $xy$  a figure  $\Phi$ , whose boundary consists of the segments  $[0, x(0)]$  of the  $x$ -axis,  $[0, 1/b]$  of the  $y$ -axis, the segment  $y = 1/b$ ,  $0 \leq x \leq x(0) - 1/b$ , and the arc of the circle*

$$y^2 + [x - x(0) + 1/b]^2 = (1/b)^2, \quad y \geq 0, \quad x \geq x(0) - 1/b.$$

Then the interior of  $\Phi$  contains no points of the curve  $x = x(y)$ .

3. The following two theorems directly lead to Theorem 1.

**Theorem 2.** *If a convex surface  $S$  has Gaussian curvature  $k$  satisfying the inequalities  $a^n \leq k \leq b^n$ , then for the diameter  $D$  of such a surface the estimate*

$$D \leq 2B + \frac{\nu_{n+1}}{\nu_n} B \left[ \left( \frac{A}{B} \right)^{(n+1)^2} - 1 \right],$$

is valid, where  $A = 1/a$ ,  $B = 1/b$ ,  $\nu_n$  is the volume of the unit  $n$ -dimensional ball.

**Proof.** Let the diameter be equal to  $D$  and lie on the  $x$ -axis. With the aid of Minkowski symmetrizations (<sup>(1)</sup>, p. 103) and a passage to the limit, one can

obtain from  $S$  a surface of revolution  $S^*$ ; moreover, one can arrange that the function  $y(x)$  be even. A suitable approximation allows us to assume, without loss of generality, that  $y(x)$  satisfies all the conditions of Lemmas 1 and 2. As is known ((<sup>2</sup>, Theorem 5)), the Gaussian curvature of  $S^*$  also does not exceed  $b^n$ , and  $D \geq 2B$  ((<sup>2</sup>, p. 5)). Therefore Lemma 2 may be applied to  $y(x)$ ; denoting by  $V_\Phi$  the volume of the body formed by rotating the figure  $\Phi$ , we have, for the volume bounded by  $S^*$ , the inequality

$$V_{S^*} \geq 2V_\Phi = \varkappa_{n+1}B^{n+1} + 2\varkappa_n[x(0) - B]B^n. \quad (1)$$

By means of Minkowski symmetrizations and passages to the limit, one can obtain from the surfaces  $S$  and  $S^*$  one and the same sphere  $S_0$ . By the Brunn-Minkowski inequality the volume  $V_{S_0} \geq V_{S^*}$ ; on the other hand,

$$V_{S_0} = \varkappa_{n+1}H_{\text{cp}}^{n+1}, \quad \text{where } H_{\text{cp}} = \frac{1}{\omega_{n+1}} \int_{\Omega^{n+1}} H d\omega; \quad (2)$$

here  $\Omega^{n+1}$  is the unit  $(n+1)$ -dimensional sphere,  $dS$  is its area element,  $\omega_{n+1}$  is the area of  $\Omega^{n+1}$ , and  $H$  is the support function of  $S$ . Comparing (1) with (2), we obtain

$$\varkappa_{n+1}H_{\text{cp}}^{n+1} \geq 2V_\Phi. \quad (3)$$

For the volume  $V_S$  bounded by the original surface  $S$ , we have

$$V_S = \frac{1}{n+1} \int_{\Omega^{n+1}} \frac{1}{k} H d\omega \geq \frac{1}{n+1} B^n \int_{\Omega^{n+1}} H d\omega = \frac{1}{n+1} \omega_{n+1} B^n H_{\text{cp}}. \quad (4)$$

From (3) and (4) we find that

$$V_S^{n+1} \geq \varkappa_{n+1}^n B^{n(n+1)} 2V_\Phi. \quad (5)$$

Finally, by Theorem 3 of (<sup>2</sup>),

$$V_S \leq \varkappa_{n+1} A^{n+1}, \quad (6)$$

and from (1), (5), and (6) we obtain the required estimate.

**Theorem 3.** If a convex surface  $S$  has Gaussian curvature  $k$  satisfying the inequality  $a^n \leq k \leq b^n$ , then for the width  $\Delta$  of such a surface the estimate

$$\Delta \geq 2A - (n+1) \frac{\varkappa_{n+1}}{\varkappa_n} A \left[ 1 - \left( \frac{B}{A} \right)^{n+1} \right].$$

**Proof.** We may assume that  $\Delta < 2A$ . Take as the  $x$ -axis a line perpendicular to the supporting planes of  $S$ , the distance between which is equal to  $\Delta$ . By means of Steiner symmetrization and a limiting passage, from  $S$  we obtain a

surface of revolution  $S^*$  with even function  $y(x)$ ; one may assume that  $y(x)$  satisfies all the conditions of Lemma 1, with  $x'(0) \leq 0$ . According to Theorem 4 ([2], Theorem 2), the Gaussian curvature of  $S^*$  is not less than  $a^n$ . By Lemma 1, the curve  $x = x(y)$  lies no higher than the circular arc

$$x = \sqrt{A^2 - y^2} + x(0) - A, \quad 0 \leq y \leq \sqrt{A^2 - [A - x(0)]^2}.$$

Denote by  $V_c$  the volume of the spherical segment bounded by the surface of revolution whose directrix is the indicated circular arc. The volume  $V_{S^*} = V_S$ , whence

$$V_S \leq 2V_c = \kappa_{n+1}A^{n+1} - \frac{2}{n+1}\kappa_n A^n [A - x(0)]. \quad (7)$$

As A. I. Fet proved ([2], Theorem 3),

$$V_S \geq \kappa_{n+1}B^{n+1}, \quad (8)$$

which together with (7) gives the required estimate.

4. Let us make an essential remark concerning Theorems 2 and 3. The conclusion of Theorem 2 (respectively, Theorem 3) remains valid if the condition  $k \geq a^n$  ( $k \leq b^n$ ) is replaced by the requirement

$$V_S \leq \kappa_{n+1}A^{n+1} \quad (V_S > \kappa_{n+1}B^{n+1}).$$

Indeed, the replaced conditions were used only to prove inequalities (6) and, respectively, (8).

5. In order to derive Theorem 1 from Theorems 2 and 3, it remains to show that the order of the estimate cannot be improved; for this it is enough to consider spheres of curvature  $1 \pm \varepsilon$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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