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Abstract

Full Text

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ON THE EMBEDDING OF CROSSED PRODUCTS IN DIVISION RINGS

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In this note we generalize the well-known Mal' cev-Neumann theorem on the embedding of group algebras of ordered groups in division rings, as well as Ikeda' s result ([1]) stating that the crossed product (G, D, ρ, σ) of an RN^*-group G with factors without twisting and a division ring D, where σ is a mapping with trivial kernel, has a division ring of right quotients. (For the definition of a crossed product and the necessary notation, see ([3]).)

Lemma 1. Let (G, D, ρ, σ) be the crossed product of a group G and a ring D; let H be an invariant subgroup of the group G. If the subring (H, D, ρ, σ) has a division ring of right quotients T, then the mapping

$$xy^{-1} \to t_g^{-1} xy^{-1} t_g = t_g^{-1} x t_g \, (t_{gyt} g^{-1})^{-1},$$

where $x, y \in (H, D, \rho, \sigma)$, $g \in G$, is an automorphism of the division ring T.

The assertion of the lemma follows immediately from the fact that the mapping $x \to t_g^{-1} x t_g$ is an automorphism of the ring (H, D, ρ, σ).

Lemma 2. Let (G, D, ρ, σ) be the crossed product of a group G and a ring D, containing no zero divisors, and let H be an invariant subgroup of the group G such that G/H is an abelian group with a finite number of generators. If there exists a ring of right quotients L of the ring (G, D, ρ, σ) relative to the subring (H, D, ρ, σ), then the ring L is Noetherian.

Proof. Let

$$G/H = \{\bar{d}_1\} \times \cdots \times \{\bar{d}_t\} \times \bar{H}_1$$

and

$$\bar{K} = \{\bar{d}_1\} \times \cdots \times \{\bar{d}_{t-1}\} \times \bar{H}_1,$$

where $\bar{H}_1$ is a finite abelian group, $\{\bar{d}_i\}$ is an infinite cyclic group, $i = 1, \dots, t$; $K(H_1)$ is the full preimage of the group $\bar{K}(\bar{H}_1)$ in the group G, and d^n is one of the preimages of the element $\bar{d}_t^n$ in the group G, $n = \dots, -2, -1, 0, 1, 2, \dots$. Every element of the ring of right quotients L_1 of the ring (H_1, D, ρ, σ) relative to the subring (H, D, ρ, σ) can be represented in the form $\sum t_{k_i} xy^{-1}$, where k_i are representatives of the cosets of the group H_1 modulo the subgroup H, $x_i, y \in (H, D, \rho, \sigma)$. Since L_1 is a finite-dimensional vector space over the division ring of right quotients of the ring (H, D, ρ, σ), it follows that L_1 is Noetherian.

Suppose that the ring of right quotients L_2 of the ring (K, D, ρ, σ) relative to the subring (H, D, ρ, σ) is Noetherian, and let us prove that L is Noetherian.

Let I be an arbitrary right ideal of the ring L. Then, by Lemma 1, every element of I can be represented in the form

$$x = \sum_{r \le n \le s} c_n t_{d^n},$$

where $c_n \in L_2,\ c_s \neq 0$; the element c_s will be called the **leading coefficient** of the element x. The set of leading coefficients of the elements belonging to I forms a right ideal I_0 of the ring L_2. Indeed, if b_t is the leading coefficient of the element y, then $x \pm y t_h \rho^{-1}_{d^t,h}$, where h is determined from the equality $d^t h = d^s$, has leading coefficient $c_s \pm b_t$, and if $w \in L_2$, then

$$x t^{-1}_{d^s} w t_{d^s}$$

has leading coefficient c_{sw}. Since L_2 is Noetherian, I_0 has a finite basis $a_1, \dots, a_l$. Let x_i be an element of the right ideal I with leading coefficient a_i, and we may suppose that in the expression of the element x_i only t_{d^n}, $0 \le n$, occur, and that x_i has leading term $a_{it^m_d}$ (m is the same for all i).

Denote by I_1 the right ideal of the ring L generated by the elements $x_1, \dots, x_l$. If x is an arbitrary element of the ideal I, then there exists an element d^k such that, in the expression of the element $y = x t_{d^k}$, only the t_{d^n}, $n \ge 0$, occur; and suppose that $\tilde{a}_p$ is the leading coefficient of the element y. Then $\tilde{a}_p = \sum a_i y_i$, where $y_i \in L_2$, and if $p \ge m$, then

$$y - \sum x_i t^{-1}_{d^m} y_i t_{d^p}$$

is an element of the right ideal I, in whose expression only the basis elements t_{d^n}, $0 \le n \le p-1$, occur. Repeating this argument, we obtain that $y \equiv v \pmod{I_1}$, and in the expression of v only t_{d^n}, $0 \le n \le m-1$, occur. The set I_2 of elements of the ideal I which are linear combinations of the elements t_{d^n}, $0 \le n \le m-1$, forms a right L_2-module. This module has a finite L_2-basis $x_{l+1}, \dots, x_r$, since the set of leading coefficients is again a right ideal of the ring L_2, and the preceding arguments may be repeated. If I_3 is the right ideal of the ring L generated by $x_1, \dots, x_r$, then $y t^{-1}_{d^k} = x \in I_3$. Consequently, $I_3 = I$. The lemma is proved.

Theorem. *Suppose that the group G has a normal divisor H such that G/H is an ordered group, and H has an ascending normal series whose factors are locally finite over their centers. If (G, D, ρ, σ) is an arbitrary crossed product of the group G and the skew field D, and the subring (H, D, ρ, σ) contains no zero divisors, then (G, D, ρ, σ) can be embedded in a skew field, and the subring (H, D, ρ, σ) has a skew field of right fractions.*

Proof. Let

$$1 = H_0 \subset H_1 \subset \cdots \subset H_\tau = H$$

be an ascending normal series of the group H, with the factors H_{i+1}/H_i locally finite groups over their centers Z_{i+1}/H_i, $0 \le i < \tau$. We shall show that the ring (H, D, ρ, σ) has a skew field of right fractions. Since for the ring (H_0, D, ρ, σ) the induction hypothesis is satisfied, we may assume that the ring $(H_\gamma, D, \rho, \sigma)$ has a skew field of right fractions L_γ for all $\gamma < \alpha$, and these skew fields are embedded in one another.

If α is a limit ordinal, then the set-theoretic union of all L_γ, $\gamma < \alpha$, is a skew field of right fractions L_α of the ring $(H_\alpha, D, \rho, \sigma)$. If there exists $\alpha - 1$, then there exists a skew field of right fractions $L_{\alpha-1}$ of the ring $(H_{\alpha-1}, D, \rho, \sigma)$.

Consider the set S of all finite sums

$$\sum t_{k_i} x_i y_i^{-1},$$

where $x_i, y_i \in (H_{\alpha-1}, D, \rho, \sigma)$, $k_i \in \Pi(Z_\alpha/H_{\alpha-1})$ are representatives of the cosets of the group Z_α by the subgroup $H_{\alpha-1}$. If $x, y, v, w \in (H_{\alpha-1}, D, \rho, \sigma)$, $k, k_1 \in \Pi(Z_\alpha/H_{\alpha-1})$, and

$$t_{k_1}^{-1} v t_{k_1} \, (t_{k_1}^{-1} w t_{k_1})^{-1} x y^{-1} = x_1 y_1^{-1}$$

in the skew field $L_{\alpha-1}$, then, by Lemma 1, multiplication may be defined in the set S as follows:

$$t_k v w^{-1} t_{k_1} x y^{-1} = t_{k_2} (t_h \rho_{k_2,h}^{-1} \rho_{k,k_1} \, x_1 y_1^{-1}),$$

where $kk_1 = k_2 h$, $k_2 \in \Pi(Z_\alpha/H_{\alpha-1})$, and $h \in H_{\alpha-1}$. Then, if we suppose that

$$\sum t_{k_i} x_i y_i^{-1} = \sum t_{k_i} v_i w_i^{-1},$$

then this holds if and only if $x_i y_i^{-1} = v_i w_i^{-1}$ for all i; with respect to the natural coordinatewise addition and the multiplication introduced, S will be a ring. Every element of the ring $(Z_\alpha, D, \rho, \sigma)$ can be represented in the form $\sum t_{k_i} x_i$, where $x_i \in (H_{\alpha-1}, D, \rho, \sigma)$, $k_i \in \Pi(Z_\alpha/H_{\alpha-1})$, and $(Z_\alpha, D, \rho, \sigma)$ will be a subring of S. Further, since in the skew field $L_{\alpha-1}$ one can reduce to a common denominator, S is the ring of right fractions of the ring (Z, D, ρ, σ) with respect to the subring $(H_{\alpha-1}, D, \rho, \sigma)$, and therefore it contains no zero divisors. We shall show that the ring S satisfies Ore' s condition. Let $xy^{-1}, vw^{-1} \in S$ and

$$x = \sum t_{k_i} y_i, \qquad v = \sum t_{l_i} w_i,$$

where $y, y_i, w, w_i \in (H_{\alpha-1}, D, \rho, \sigma)$, $k_i, l_i \in \Pi(Z_\alpha/H_{\alpha-1})$. Denote by K the subgroup of the group H_α generated by the elements $k_1, \dots, k_n, l_1, \dots, l_m$ and by the group $H_{\alpha-1}$. Then $K/H_{\alpha-1}$ is an abelian group with a finite number of generators, and xy^{-1}, vw^{-1} are elements of the skew field of right fractions S_1 of the ring (K, D, ρ, σ) with respect to—

relative to the subring $(H_{\alpha-1}, D, \rho, \sigma)$. By Lemma 2, the ring S_1 is nonzero, and in such rings, as Goldie showed (see (4)), the Ore condition is satisfied. Consequently, the ring $(Z_\alpha, D, \rho, \sigma)$ has a skew field of right quotients S_2.

Similarly one can show the existence of a ring of right quotients S_3 of the ring $(H_\alpha, D, \rho, \sigma)$ with respect to the subring $(Z_\alpha, D, \rho, \sigma)$. The ring S_3 contains no zero divisors and is a skew field. Indeed, if

$$x = \sum_{i=1}^{n} t_{k_i} x_i y_i^{-1},$$

where $x_i y_i^{-1} \in S_2,\ k_i \in \Pi(H_\alpha/Z_\alpha)$, and M is the subgroup of the group H_α generated by Z_α and the elements $k_1, \dots, k_n$, then the factor group M/Z_α is finite and x is an element of the ring of right quotients S_4 of the ring (M, D, ρ, σ) with respect to the subring $(Z_\alpha, D, \rho, \sigma)$. The ring S_4 is a finite-dimensional vector space over a skew field and has no zero divisors. Hence x is an invertible element and S_3 is a skew field of right quotients of the ring $(H_\alpha, D, \rho, \sigma)$. Thus the subring (H, D, ρ, σ) has a skew field of right quotients L.

Construct, as above, a ring of right quotients L_1 of the ring (G, D, ρ, σ) with respect to the subring (H, D, ρ, σ). It is clear that $L \subseteq L_1$. Then every element of the ring L_1 can be written in the form of a finite sum $\sum t_{k_i} x_i y_i^{-1}$, where $x_i, y_i \in (H, D, \rho, \sigma),\ k_i \in \Pi(G/H)$. Denote by $\bar{k}_i$ the coset k_{iH}. We shall call a formal infinite sum $\sum t_{k_i} x_i y_i^{-1}$ an L-series if the set of elements $\bar{k}_i$ for which the coefficients $x_i y_i^{-1} \neq 0$ standing at t_{k_i} is well ordered in decreasing order in the sense of the prescribed ordering of the group G/H. Adding L-series by the usual rules, we again obtain an L-series. In order to multiply two formal sums $\sum t_{k_i} a_i,\ \sum t_{h_i} b_i$, where $a_i, b_i \in L,\ k_i, h_i \in \Pi(G/H)$, one must multiply each term of the first sum by each term of the second, as defined in L_1, write the resulting products in the form of a formal sum, and collect like terms. It is easy to check that the product of two L-series is defined and is an L-series, and therefore the set of L-series will be a ring. We show that each of its elements is invertible. For this it is enough to construct an inverse element for an L-series of the form $t_1 \rho_{1,1}^{-1} + \sum t_{k_i} a_i = t_1 \rho_{1,1}^{-1} + u$. By A. I. Mal' cev' s lemma (2), the series $t_1 \rho_{1,1}^{-1} - u + u^2 - u^3 + \cdots$ makes sense and is an L-series. Then

$$(t_1 \rho_{1,1}^{-1} + u)(t_1 \rho_{1,1}^{-1} - u + u^2 - u^3 + \cdots) = t_1 \rho_{1,1}^{-1}.$$

The theorem is proved.

Let us note that if the group H is right-orderable (and RN^*-groups with torsion-free factors have this property), then, by (3), the ring (H, D, ρ, σ) contains no zero divisors.

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Note: Figure translations are in progress. See original paper for figures.