MATHEMATICS

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1963

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Abstract

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MATHEMATICS

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ON RINGS WITH INJECTIVE CYCLIC MODULES

(Presented by Academician A. I. Malʼ cev on 29 VI 1962)

It is known that all left modules over a ring R are injective if and only if R is semisimple in the classical sense ((2), p. 27, Theorem 4.2). In the present note the following is proved.

Theorem. *If all left cyclic modules over a ring R are injective, then R is semisimple in the classical sense.*

In the arguments below, without special references, we use the basic notions of homological algebra ((2), Ch. I) and of the theory of regular rings ((3), §2; (8), Ch. VI, §3; (9), Part II, Ch. II). All rings considered are assumed to be associative and to have an identity, and all modules considered are unitary. We shall denote the right and left annihilators of a subset A of a ring R by A^r and A^l, respectively. A system Ω of nonzero idempotents of a ring R will be called **orthogonal** if $\varepsilon\delta = 0$ for any distinct $\varepsilon, \delta \in \Omega$. We denote by $\mathfrak{S}(M)$ the partially ordered set of cyclic submodules of a left module M. If $\mathfrak{A}$ is a subset of $\mathfrak{S}(M)$, then the least upper bound of the set $\mathfrak{A}$ in the partially ordered set $\mathfrak{S}(M)$ (if it exists) will be denoted by $\sup \mathfrak{A}$. For a regular ring R, the partially ordered set $\mathfrak{S}(M)$ turns out to be a Dedekind structure with complements. If this structure is complete, then the ring R is called **complete**. If, in addition, for every increasing chain $\{T_\alpha\}$ from $\mathfrak{S}(R)$ and every $T \in \mathfrak{S}(R)$ one has

$$(\sup T_\alpha) \cap T = \sup(T_\alpha \cap T),$$

then the ring R is called **left continuous** ((11), p. 599; (3), p. 111). Finally, note that from the theorem proved here there follows the coincidence of the classes IIIo, III , and III introduced by the author in his report at the IV All-Union Mathematical Congress (4).

Lemma 1. *Let R be a regular ring, $\mathfrak{A} \subseteq \mathfrak{S}(R)$, and $R\alpha = \sup \mathfrak{A}$. If $\beta \in T^r$ for all $T \in \mathfrak{A}$, then $\alpha\beta = 0$.*

Indeed, $\beta R \subseteq T^r$ by assumption. In view of (3), p. 20, Proposition 13 b, $(\beta R)^l \supseteq T$ for all $T \in \mathfrak{A}$. But then $(\beta R)^l \supseteq R\alpha$. Applying (3), p. 20, Proposition 13 , we arrive at $\beta R \subseteq (R\alpha)^r$, whence $\alpha\beta = 0$.

Lemma 2. *Let R be a complete ring, $\mathfrak{A} = \{R\alpha_1, R\alpha_2, \dots\}$ a countable independent ((3), pp. 96-97) subset of $\mathfrak{S}(R)$, $R\alpha_i \neq 0$, and $\sup \mathfrak{A} = R$. Then there exists an orthogonal system of idempotents $\{\varepsilon_1, \varepsilon_2, \dots\}$ such that $R\alpha_i = R\varepsilon_i$.*

For the proof put $T_0 = R$ and

$$T_k = \sup_{k<i<\infty} R\alpha_i \quad (k = 1, 2, \dots).$$

Let $\varepsilon_0 = 0$, $\delta_0 = 1$. Suppose that systems of idempotents $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n$ and $\delta_0, \delta_1, \dots, \delta_n$ have been constructed such that the system $\{\varepsilon_1, \dots, \varepsilon_n\}$ is orthogonal, $R\varepsilon_i = R\alpha_i$ and $R\delta_i = T_i$ for $i = 0, 1, \dots, n$, and, moreover, for all $k = 1, \dots, n$ one has

$$(\varepsilon_1 + \dots + \varepsilon_k)\delta_k = \delta_k(\varepsilon_1 + \dots + \varepsilon_n) = 0,$$
$$\varepsilon_1 + \dots + \varepsilon_k + \delta_k = 1.$$

Let us write*

$$R = R\varepsilon_1 \dotplus \dots \dotplus R\varepsilon_n \dotplus R\alpha_{n+1} \dotplus T_{n+1}.$$

In view of (3), p. 20, Proposition 13, a, there is an orthogonal system of idempotents $\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n, \bar{\varepsilon}_{n+1}, \bar{\delta}_{n+1}$ such that $R\varepsilon_i = R\bar{\varepsilon}_i$ $(i = 1, \dots, n)$, $R\varepsilon_{n+1} = R\alpha_{n+1}$, $R\bar{\delta}_{n+1} = T_{n+1}$, and $1 = \bar{\varepsilon}_1 + \dots + \bar{\varepsilon}_n + \bar{\varepsilon}_{n+1} + \bar{\delta}_{n+1}$. Since $R(\varepsilon_{n+1} + \delta_{n+1}) = R\bar{\varepsilon}_{n+1} \dotplus R\bar{\delta}_{n+1} = T_n$, by the uniqueness indicated in (3), p. 20, Proposition 13, a, $\bar{\varepsilon}_i = \varepsilon_i$ $(i = 1, \dots, n)$.

Lemma 3. *Let R be a regular ring; I its left ideal; $M = R/I$; T^* the image of the left ideal T under the natural homomorphism of R onto M; $\{\varepsilon_i\}$ a system of idempotents of the ring R; $\varepsilon^2 = \varepsilon$; $(R\varepsilon_i)^* \ll (R\varepsilon)^*$ for all i. Then, if in R there exists an idempotent σ such that: a) $\varepsilon\sigma \notin I$; b) $\varepsilon_i\sigma \in I$ for all i; c) $I\sigma \subseteq I$, then either there is an idempotent $\theta \in R$ such that $(R\varepsilon)^* >> (R\theta)^* >> (R\varepsilon_i)^*$ for all i, or else $\mathfrak{S}(M)$ is not a structure**.*

For the proof put $\alpha = \varepsilon - \varepsilon\sigma$. Since $(R\varepsilon_i)^* \ll (R\varepsilon)^*$, we have $\varepsilon_i = \lambda_i\varepsilon + x_i$, where $\lambda_i \in R$, $x_i \in I$. Moreover, $\lambda_i\alpha = \lambda_i\varepsilon - \lambda_i\varepsilon\sigma$. Hence

$$\varepsilon_i = \lambda_i\alpha + \lambda_i\varepsilon\sigma + x_i.$$

Multiplying on the right by σ and taking b) and c) into account, we obtain

$$\lambda_i\varepsilon\sigma = \varepsilon_i\sigma - x_i\sigma \in I.$$

Consequently, $(R\varepsilon_i)^* \ll (R\alpha)^*$ for all i. Therefore, putting

$$(R\theta)^* = \inf\{(R\alpha)^*, (R\varepsilon)^*\},$$

where $\theta^2 = \theta$, we shall have

$$(R\varepsilon)^* \geq (R\theta)^* \geq (R\varepsilon_i)^*$$

for all i. If $(R\varepsilon)^* = (R\theta)^*$, then

$$\varepsilon = \lambda\theta + x = \lambda\mu\alpha + y,$$

where $\lambda, \mu \in R$, $x, y \in I$. Multiplying on the right by σ and taking c) into account, we arrive at $\varepsilon\sigma = x\sigma \in I$, which contradicts a). Thus $(R\varepsilon)^* > (R\theta)^*$.

Lemma 4. *If Ω is an orthogonal system of idempotents of a left continuous ring R, then the system $\{R\varepsilon;\ \varepsilon \in \Omega\}$ is independent in the structure $\mathfrak{S}(R)$.*

The validity of this lemma is not difficult to derive with the aid of (^{3}), p. 113, Proposition 75. It should be noted that, although this proposition is formulated for continuous geometry, in its proof only the properties indicated in Proposition 74 are used.

Lemma 5. *If $\{\varepsilon_1, \varepsilon_2, ...\}$ is a countable orthogonal system of idempotents of a left continuous ring R, $\sup R\varepsilon_i = R$, $\alpha \in R$, and $\alpha\varepsilon_i = 0$ for all i, then $\alpha = 0$.*

Indeed, if $\beta \in R\alpha \cap (R\varepsilon_1 + ... + R\varepsilon_m)$, then

$$\beta = \xi\alpha(\varepsilon_1 + ... + \varepsilon_m) = 0.$$

Since

$$\sup_i R\varepsilon_i = \sup_k (R\varepsilon_1 + ... + R\varepsilon_k),$$

it follows from continuity that

$$R\alpha = R\alpha \cap \sup_i R\varepsilon_i = 0.$$

Lemma 6. *Let R be a left continuous ring, Ω an orthogonal system of idempotents from R, $\sup_{\varepsilon\in\Omega} R\varepsilon = R$, $\Xi \subseteq \Omega$. Then there exists an idempotent σ such that $\sup_{\varepsilon\in\Xi} R\varepsilon = R\sigma$ and $\sigma\varepsilon = \varepsilon$ for all $\varepsilon \in \Xi$.*

Indeed, put

$$U = \sup_{\varepsilon\in\Xi} R\varepsilon \quad \text{and} \quad V = \sup_{\varepsilon\in\Omega\setminus\Xi} R\varepsilon.$$

In view of Lemma 4,

$$R = U \dotplus V.$$

From (^{3}), p. 20, Proposition 13, a, it follows that there exist idempotents σ and τ such that

$$U = R\sigma, \quad V = R\tau, \quad \sigma\tau = \tau\sigma = 0, \quad 1 = \sigma + \tau.$$

If $\varepsilon \in \Xi$, then $\delta\varepsilon = 0$ for all $\delta \in \Omega \setminus \Xi$. Therefore, from Lemma 1 it follows that $\tau\varepsilon = 0$. Hence

$$\varepsilon = \sigma\varepsilon + \tau\varepsilon = \sigma\varepsilon.$$

Lemma 7. *Let Ω be a countable orthogonal set of idempotents of a left continuous ring R, $\sup_{\rho\in\Omega} R\rho = R$, $I = \sum_{\rho\in\Omega} R\rho$, and $M = R/I$. Then either in the* *partially ordered set* $\mathfrak{S}(M)$ *there exists a subset having no least upper bound, or else* $\mathfrak{S}(M)$ *is not a structure*[**].[*]

For the proof, renumber the idempotents from Ω by pairs of natural numbers:

$$\Omega = \{\varepsilon_{ij}\}.$$

In view of Lemma 6 there exist such idempotents ε_i, that $R\varepsilon_i = \sup\limits_k R\varepsilon_{ik}$ and

$$\varepsilon_i \varepsilon_{ik} = \varepsilon_i, \quad \text{for all } k. \tag{*}$$

Suppose that an idempotent $\varepsilon \in R$ is such that $(R\varepsilon)^* \geq (R\varepsilon_i)^*$ for all i (the notation of Lemma 3 is used). Then

$$\varepsilon_i = \lambda_i \varepsilon + x_i, \tag{**}$$

where $\lambda_i \in R$, $x_i \in I$. Choose a number $k(i)$ such that $x_i \varepsilon_{ik(i)} = 0$. If $\varepsilon\varepsilon_{ik(i)} = 0$, then from $(*)$ and $(**)$ it follows that

$$\varepsilon_{ik(i)} = \varepsilon_i \varepsilon_{ik(i)} = \lambda_i \varepsilon \varepsilon_{ik(i)} + x_i \varepsilon_{ik(i)} = 0.$$

Consequently,

$$\varepsilon\varepsilon_{ik(i)} \neq 0. \tag{***}$$

Using Lemma 6 again, choose an idempotent σ such that $R\sigma = \sup R\varepsilon_{ik(i)}$ and $\sigma\varepsilon_{ik(i)} = \varepsilon_{ik(i)}$. If $\varepsilon\sigma \in I$, then $\varepsilon\varepsilon_{sk(s)} = \varepsilon\sigma\varepsilon_{sk(s)} = 0$ for some number s. This, however, contradicts $(***)$, and therefore

$$\varepsilon\sigma \notin I.$$

Let, further, $R\alpha_i = \sup\limits_{s \neq k(i)} R\varepsilon_{is}$. If $s \neq k(i)$, then, in view of Lemma 1,

$$\varepsilon_{is}\sigma\varepsilon_{pq} = \begin{cases} \varepsilon_{is} \cdot 0 = 0, & \text{if } q \neq k(p), \\ \varepsilon_{is}\varepsilon_{pk(p)} = 0, & \text{if } q = k(p). \end{cases}$$

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[*] Dots over the plus signs mean that the summands are independent (see (3), pp. 14–15).

[**] Cf. (5), p. 453, Theorem 4.1.

[***] Cf. (5), p. 454, Theorem 4.2.

But then Lemma 5 gives $\varepsilon_{is}\sigma = 0$. After this Lemma 1 gives $\alpha_i\sigma = 0$. Since $R\varepsilon_i = R\varepsilon_{ik(i)} + R\alpha_i$, we have $\varepsilon_i = \xi\varepsilon_{ik(i)} + \eta\alpha_i$. Multiplying on the right by σ, we obtain

$$\varepsilon_i\sigma = \xi\varepsilon_{ik(i)} \in I.$$

Hence $\varepsilon_{ik}\sigma = \varepsilon_{ik}\varepsilon_i\sigma \in I$, and therefore

$$I\sigma \subseteq I.$$

Thus the conditions of Lemma 3 are satisfied. Consequently, $(R\varepsilon)^* \geq (R\theta)^* \geq (R\varepsilon_i)^*$ for some idempotent $\theta \in R$. In view of the arbitrariness of ε, this shows that the set $\{(R\varepsilon_i)^*\}$ has no exact upper bound in $\mathfrak{S}(M)$.

Lemma 8. *A direct summand of a left cyclic R-module is a cyclic module.*

Indeed, if $Rm = A + B$, then $m = a + b$, where $a \in A$, $b \in B$. If $x \in A$, then $x = \xi m = \xi a + \xi b$. Hence $x - \xi a = \xi b \in A \cap B$. Consequently, $x = \xi a$, i.e. $A = Ra$.

Corollary. *An injective submodule of a left cyclic R-module is always cyclic.*

Lemma 9. *If a left cyclic R-module M and all its cyclic submodules are injective, then $\mathfrak{S}(M)$ is a complete structure with complements.*

For the proof choose $\mathfrak{A} \subseteq \mathfrak{S}(M)$ and put $A = \sum\limits_{A_i \in \mathfrak{A}} A_i$. In view of the corollary to Lemma 8, the minimal injective extension $\hat{A}$ of the module A ([6], pp. 77–78; [11], p. 598) belongs to $\mathfrak{S}(M)$. If $B \in \mathfrak{S}(M)$ and $A_i \leq B$ for all i, then $\hat{A} \leq B$ in view of the injectivity of the module B. Thus,

$$\hat{A} = \sup_{A_i \in \mathfrak{A}} A_i.$$

The existence of complements follows from Lemma 8.

Lemma 10. *If a left continuous ring R does not satisfy the minimality condition for principal left ideals, then it contains a countable orthogonal system of idempotents Ω such that $R = \sup\limits_{\varepsilon \in \Omega} R\varepsilon$.*

Indeed, let $R = T_0 > T_1 > T_2 > \cdots$ be an infinite descending chain of principal left ideals of the ring R. Let $T_i = T_{i+1} + U_i$.

Then

$$(U_m + U_{m+1} + \cdots + U_{s+1}) \cap U_s \leqslant T_{s+1} \cap U_s = 0.$$

In view of (3), p. 14, Proposition 2, every finite set of ideals U_i is independent. By (3), p. 113, Proposition 75, the whole set $\{U_1, U_2, ...\}$ is also independent. Let $U = \sup U_i$ and $R = U \dotplus V$. In view of (3), p. 114, Proposition 77, the set $\mathfrak{A} = \{V, U_1, U_2, ...\}$ is independent and $\sup \mathfrak{A} = R$. It remains to apply Lemma 2.

Proof of the theorem. Obviously, R is self-injective on the left. It follows from Lemma 9 that R is regular. But then R is a semisimple ring in the sense of Jacobson ((7), p. 305, Theorem 8). Therefore, from (11), p. 599, Theorem 1, it follows that the ring R is continuous on the left. If the theorem is false, then the ring R does not satisfy the minimality condition for principal left ideals ((1), p. 135, Theorem 1; incidentally, this fact is not difficult to derive directly). In view of Lemma 10, there is in R such an orthogonal system of idempotents $\{\varepsilon_1, \varepsilon_2, ...\}$ that $R = \sup R\varepsilon_i$. Let $I = \sum R\varepsilon_i$. From Lemma 7 it follows that the partially ordered set $\mathfrak{S}(R/I)$ is not a complete lattice. This, however, contradicts Lemma 9.

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Received
21 VI 1962

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