MECHANICS

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Abstract

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ON THE STABILITY OF THE MOTION OF A SHAFT

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Consider a vertical weightless shaft on which a pulley, having the form of a thin disk, is mounted centrally (Fig. 1). Let the principal central moments of inertia of the pulley with respect to axes lying in its plane be unequal. We shall show that, at certain angular velocities of rotation, parametric resonance may arise in the system under consideration.

Taking the unperturbed motion of the shaft to be its uniform rotation about the axis, we shall write the differential equations of the perturbed motion, determined by the functions $\xi(t)$, $\eta(t)$, $\gamma(t)$, $\beta(t)$. Here ξ and η are the displacements of the center of mass of the pulley in the directions of the axes $O\xi$ and $O\eta$; the angles γ and β are shown in Fig. 2.

We shall regard the deformations of the elastic shaft as small and neglect products of small quantities in comparison with the quantities themselves. The displacement of the point O_1 in the direction of the ζ axis will be taken to be zero.

We express the angular-velocity vector of the pulley's rotation $\vec{\Omega} = \mathbf{i}\dot{\gamma} + \mathbf{n}_2\dot{\beta} + \mathbf{k}'\dot{\varphi}$ in terms of its projections on the moving axes:

$$\vec{\Omega} = \mathbf{i}'(\dot{\gamma}\sin\omega t - \dot{\beta}\cos\omega t) + \mathbf{j}'(\dot{\gamma}\cos\omega t + \dot{\beta}\sin\omega t) + \mathbf{k}'\omega, \tag{1}$$

where $\omega = \dot{\varphi}$. The momentum of the system is

$$\mathbf{Q} = im\dot{\xi} + jm\dot{\eta}, \tag{2}$$

m being the mass of the pulley.

The moment of momentum of the pulley with respect to the origin of the fixed coordinate system $O\xi\eta\zeta$ is determined by means of the known formula:

$$\mathbf{K} = \mathbf{r}_0 \times \mathbf{Q} + \sum_1^n m_s\mathbf{r}_s \times (\bar{\Omega} \times \mathbf{r}_s), \tag{3}$$

Fig. 1

Figure 1: Fig. 1

where $\mathbf{r}_0$ is the radius vector of the point O_1 with respect to the point O.

Assuming that the moments of inertia of the pulley with respect to the axes Ox and Oy are related by the dependence $I_y = I_x(1+\varepsilon)$, where ε is small compared with unity, and substituting (1) and (2) into (3), as a result of simple transformations we obtain:

$$\mathbf{K} = \mathbf{i}I_x\left[\dot\gamma + \varepsilon\left(\dot\gamma\cos^2\omega t + \dot\beta\sin^2\omega t/2\right) + 2\omega\beta\right] + \mathbf{j}I_x\left[\dot\beta + \varepsilon\left(\dot\beta\sin^2\omega t + \dot\gamma\sin 2\omega t/2\right) - 2\omega\gamma\right] + \mathbf{k}I_z\omega.$$

The components of the forces and moments applied to the pulley by the elastic shaft are:

$$P_\xi = -\frac{12(3l_2+l_1)EI}{(3l_2+4l_1)l_2^3}\,\xi + \frac{6(3l_2+2l_1)EI}{(3l_2+4l_1)l_2^2}\,\beta,$$

Fig. 1

$$P_\eta = -\frac{12(3l_2+l_1)EI}{(3l_2+4l_1)l_2^3}\eta - \frac{6(3l_2+2l_1)EI}{(3l_2+4l_1)l_2^2}\gamma,$$

$$M_\eta = \frac{6EI(3l_2+2l_1)}{(3l_2+4l_1)l_2^2}\xi - \frac{12(l_2+l_1)EI}{(3l_2+4l_1)l_2}\beta,$$

$$M_\xi = -\frac{6(3l_2+2l_1)EI}{(3l_2+4l_1)l_2^2}\eta - \frac{12(l_2+l_1)EI}{(3l_2+4l_1)l_2}\gamma$$

can readily be found by means of Mohr' s method. Here EI is the flexural rigidity of the shaft.

Substituting $K, Q, P_\xi, P_\eta, M_\xi, M_\eta$ into the equations expressing the theorems on the change of momentum and angular momentum,

$$dQ/dt = P, \qquad dK/dt = M,$$

we find

$$im\ddot\xi + jm\ddot\eta = iP_\xi + jP_\eta,$$

Fig. 2

Figure 2: Fig. 2

$$\begin{aligned} iI_x[\ddot\gamma + \varepsilon(\ddot\gamma\cos^2\omega t - \dot\gamma\omega\sin 2\omega t/2 + \dot\beta\sin 2\omega t/2 + \dot\beta\omega\cos 2\omega t) \\ + 2\omega\dot\beta] + jI_x[\ddot\beta + \varepsilon(\ddot\beta\sin^2\omega t + \dot\beta\omega\sin 2\omega t/2 + \ddot\gamma\sin 2\omega t/2 \\ + \dot\gamma\omega\cos 2\omega t) - 2\omega\dot\gamma] = iM_\xi + jM_\eta. \end{aligned} \tag{4}$$

Fig. 2

The two vector equations (4) are equivalent to the following system of four scalar differential equations with variable coefficients:

$$\ddot\xi + \delta_1\lambda\xi - \delta_2\lambda\beta = 0,$$

$$\ddot\eta + \delta_1\lambda\eta + \delta_2\lambda\gamma = 0,$$

$$\begin{aligned} \ddot\gamma(1+\varepsilon\cos^2\tau) + \varepsilon\dot\beta\sin 2\tau/2 - \varepsilon\dot\gamma\sin 2\tau + \dot\beta(2+\varepsilon\cos 2\tau) \\ + \delta_3\lambda\eta + \delta_4\lambda\gamma = 0, \\ \ddot\beta(1+\varepsilon\sin^2\tau) + \varepsilon\dot\gamma\sin 2\tau/2 + \varepsilon\dot\beta\sin 2\tau - \dot\gamma(2-\varepsilon\cos 2\tau) \\ - \delta_3\lambda\xi + \delta_4\lambda\beta = 0, \end{aligned} \tag{5}$$

where $\tau = \omega t,\ \lambda = 1/\omega^2,\ \dot\gamma = d\gamma/d\tau,\ \dots,$

$$\delta_1 = \frac{12(3l_2 + l_1)EI}{(3l_2 + 4l_1)l_2^3 m},$$

$$\delta_2 = \frac{6(3l_2 + 2l_1)EI}{(3l_2 + 4l_1)l_2^2 m}, \qquad \delta_3 = \frac{6(3l_2 + 2l_1)EI}{(3l_2 + 4l_1)l_2^2 I_x},$$

$$\delta_4 = \frac{12(l_1 + l_2)EI}{(3l_2 + 4l_1)l_2 I_x}.$$

This system of equations is reversible ([1]). Indeed, one of the particular solutions of the system corresponding to the characteristic number ρ_k has the form:

$$\beta = e^{\frac{\tau}{\pi}\ln\rho_k}\psi_1(\tau), \qquad \gamma = e^{\frac{\tau}{\pi}\ln\rho_k}\psi_2(\tau), \dots,$$

where $\psi_1, \psi_2, \ldots$ are periodic functions of τ. When τ is replaced by $-\tau$, β by $-\beta$, and ξ by $-\xi$, the equations do not change; therefore $1/\rho_k$ is also a characteristic number of the system.

It is clear that for any real ρ_k the motion will be unstable. In the case of complex characteristic numbers, $|\rho_1| = \cdots = |\rho_4| = 1$, and stable motion of the shaft is possible ([1]).

It follows from this that, in the coordinate system ε, λ, the region of instability is separated from the region in which stable motion is possible by the curves $\lambda = \lambda(\varepsilon)$ corresponding to the π- and 2π-periodic solutions of equations (5).

To find these solutions, we shall use the method of a small para-

meter (1). Let us represent the functions ξ, η, γ, β and the parameter λ in the form

$$\begin{gathered}\xi = \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \ldots, \qquad \eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \ldots, \\ \gamma = \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \ldots, \qquad \beta = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \ldots, \\ \lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots.\end{gathered} \tag{6}$$

Substituting (6) into (5) and comparing coefficients of like powers of ε, we obtain the following systems of linear differential equations with constant coefficients:

$$\begin{aligned}\ddot{\xi}_0 + \delta_1\lambda_0\xi_0 - \delta_2\lambda_0\beta_0 = 0, &\qquad \ddot{\eta}_0 + \delta_1\lambda_0\eta_0 + \delta_2\lambda_0\gamma_0 = 0, \\ \ddot{\gamma}_0 + 2\dot{\beta}_0 + \delta_3\lambda_0\eta_0 + \delta_4\lambda_0\gamma_0 = 0, &\qquad \ddot{\beta}_0 - 2\dot{\gamma}_0 - \delta_3\lambda_0\xi_0 + \delta_4\lambda_0\beta_0 = 0;\end{aligned} \tag{7}$$

$$\begin{gathered}\ddot{\xi}_1 + \delta_1\lambda_0\xi_1 - \delta_2\lambda_0\beta_1 = -\delta_1\lambda_1\xi_0 + \delta_2\lambda_1\beta_0, \\ \ddot{\eta}_1 + \delta_1\lambda_0\eta_1 + \delta_2\lambda_0\gamma_1 = -\delta_1\lambda_1\eta_0 - \delta_2\lambda_1\gamma_0, \\ \ddot{\gamma}_1 + 2\dot{\beta}_1 + \delta_3\lambda_0\eta_1 + \delta_4\lambda_0\gamma_1 \\ = -\ddot{\gamma}_0\cos^2\tau - \ddot{\beta}_0\sin 2\tau/2 + \dot{\gamma}_0\sin 2\tau - \beta_0\cos 2\tau - \delta_3\lambda_1\eta_0 - \delta_4\lambda_1\gamma_0, \\ \ddot{\beta}_1 - 2\dot{\gamma}_1 - \delta_3\lambda_0\xi_1 + \delta_4\lambda_0\beta_1 \\ = \ddot{\beta}_0\sin^2\tau - \dot{\gamma}_0\sin 2\tau/2 - \beta_0\sin 2\tau - \dot{\gamma}_0\cos 2\tau + \delta_3\lambda_1\xi_0 - \delta_4\lambda_1\alpha_0,\end{gathered} \tag{8}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

We note that the oscillations of a circular ($\varepsilon = 0$) pulley mounted on a shaft, determined by the differential equations (7), were investigated in works ([2,3]).

To find the 2π-periodic solutions of system (7), let us represent the functions ξ_0, η_0, γ_0, and β_0 in the form:

$$\xi_0 = a_0 + \sum_1^\infty a_n \cos n\tau + a_n' \sin n\tau, \qquad \eta_0 = b_0 + \sum_1^\infty b_n \cos n\tau + b_n' \sin n\tau,$$

$$\gamma_0 = c_0 + \sum_1^\infty c_n \cos n\tau + c_n' \sin n\tau, \qquad \beta_0 = d_0 + \sum_1^\infty d_n \cos n\tau + d_n' \sin n\tau. \quad (9)$$

Substituting these expansions into (7) and comparing coefficients of like functions, we obtain, as a result of simple transformations, the following systems of homogeneous algebraic equations with respect to the coefficients of the expansions (9):

$$\begin{aligned} \delta_1\lambda_0 a_0 - \delta_2\lambda_0 d_0 = 0, &\qquad \delta_3\lambda_0 b_0 + \delta_4\lambda_0 c_0 = 0, \\ -\delta_3\lambda_0 a_0 + \delta_4\lambda_0 d_0 = 0, &\qquad \delta_1\lambda_0 b_0 + \delta_2\lambda_0 c_0 = 0; \end{aligned} \qquad (10)$$

$$\begin{gathered} (\delta_1\lambda_0 - n^2)(a_n + b_n') + \delta_2\lambda_0(c_n' - d_n) = 0, \\ \delta_3\lambda_0(a_n + b_n') + (\delta_4\lambda_0 - n^2 + 2n)(c_n' - d_n) = 0, \\ (\delta_1\lambda_0 - n^2)(a_n - b_n') - \delta_2\lambda_0(c_n' + d_n) = 0, \\ -\delta_3\lambda_0(a_n - b_n') + (\delta_4\lambda_0 - n^2 - 2n)(c_n' + d_n) = 0, \\ (\delta_1\lambda_0 - n^2)(a_n' + b_n) + \delta_2\lambda_0(c_n - d_n') = 0, \\ \delta_3\lambda_0(a_n' + b_n) + (\delta_4\lambda_0 - n^2 - 2n)(c_n - d_n') = 0, \\ (\delta_1\lambda_0 - n^2)(a_n' - b_n) - \delta_2\lambda_0(c_n + d_n') = 0, \\ -\delta_3\lambda_0(a_n' - b_n) + (\delta_4\lambda_0 - n^2 + 2n)(c_n + d_n') = 0. \end{gathered} \qquad (11)$$

Since the determinant of system (10) is different from zero, it follows that $a_0 = b_0 = c_0 = d_0 = 0$.

Observing that the determinants of systems (11) coincide pairwise, we find that either

$$\begin{vmatrix} \delta_1\lambda_0 - n^2 & \delta_2\lambda_0 \\ \delta_3\lambda_0 & \delta_4\lambda_0 - n^2 + 2n \end{vmatrix} = 0,$$

$$\lambda_0 = \lambda_0' = \frac{(\delta_1 + \delta_4)n^2 - 2n\delta_1 + \sqrt{\left[(\delta_1 + \delta_4)n^2 - 2n\delta_1\right]^2 - 4(n^4 - 2n^3)}}{2(\delta_1\delta_4 - \delta_2\delta_3)},$$

$$a_n' = -b_n, \qquad d_n' = c_n, \qquad b_n' = a_n, \qquad c_n' = -d_n;$$

or

$$\begin{vmatrix} \delta_1\lambda_0 - n^2 & -\delta_2\lambda_0 \\ -\delta_3\lambda_0 & \delta_4\lambda_0 - n^2 - 2n \end{vmatrix} = 0,$$

$$\lambda_0 = \lambda_0'' = \frac{(\delta_1+\delta_4)n^2 + 2n\delta_1 + \sqrt{\left[(\delta_1+\delta_4)n^2 + 2n\delta_1\right]^2 - 4\left(n^4 + 2n^3\right)}}{2(\delta_1\delta_4 - \delta_2\delta_3)},$$

$$b_n' = -a_n, \qquad c_n' = d_n, \qquad a_n' = b_n, \qquad d_n' = -c_n.$$

Here the negative roots λ_0 have been discarded, since in the zero approximation λ_0 is a quantity reciprocal to the square of the angular velocity.

From the expressions obtained for λ_0' and λ_0'' it follows that periodic solutions of system (7) occur only for integer n determined by the inequalities

$$\left[(\delta_1+\delta_4)^2 n^2 \pm 2n\delta_1\right]^2 \geq 4\left(n^4 \mp 2n^3\right).$$

Thus we arrive at the following result: for the angular velocity of rotation of the shaft $\omega' = \sqrt{1/\lambda_0'}$, periodic motion of the circular pulley is possible, determined by the equations

$$\begin{aligned} \xi_0 &= a_n \cos n\tau - b_n \sin n\tau, & \gamma_0 &= c_n \cos n\tau - d_n \sin n\tau, \\ \eta_0 &= b_n \cos n\tau + a_n \sin n\tau, & \beta_0 &= d_n \cos n\tau + c_n \sin n\tau; \end{aligned} \tag{12}$$

for the angular velocity $\omega'' = \sqrt{1/\lambda_0''}$ one may expect the emergence of periodic motion whose equations have the form

$$\begin{aligned} \xi_0 &= a_n \cos n\tau + b_n \sin n\tau, & \gamma_0 &= c_n \cos n\tau + d_n \sin n\tau, \\ \eta_0 &= b_n \cos n\tau - a_n \sin n\tau, & \beta_0 &= d_n \cos n\tau - c_n \sin n\tau. \end{aligned} \tag{13}$$

The presence in each solution of 4 (and not 8) constants of integration indicates that the periodic motions determined by the parameters λ_0' and λ_0'' can be realized not under arbitrary initial conditions. The possibility of such motions for $n = 1$ ("direct" and "reverse" precession) has been pointed out repeatedly in the literature ([2], [3]).

To find 2π-periodic solutions corresponding to λ_0' for $n = 1$, substitute (12) into (8):

$$\ddot{\xi}_1 + \delta_1\lambda_0'\xi_1 - \delta_2\lambda_0'\beta_1 = -\delta_1\lambda_1(a_1\cos\tau - b_1\sin\tau) + \delta_2\lambda_1(d_1\cos\tau + c_1\sin\tau),$$

$$\ddot{\eta}_1 + \delta_1\lambda_0'\eta_1 + \delta_2\lambda_0'\gamma_1 = -\delta_1\lambda_1(b_1\cos\tau + a_1\sin\tau) - \delta_2\lambda_1(c_1\cos\tau - d_1\sin\tau),$$

$$\ddot{\gamma}_1 + 2\dot{\beta}_1 + \delta_3\lambda_0'\eta_1 + \delta_4\lambda_0'\gamma_1 = (c_1\cos\tau - d_1\sin\tau)\cos^2\tau +$$

$$+(d_1\cos\tau + c_1\sin\tau)\frac{\sin 2\tau}{2} - (c_1\sin\tau + d_1\cos\tau)\sin 2\tau +$$

$$+(d_1\sin\tau - c_1\cos\tau)\cos 2\tau - \delta_3\lambda_1(b_1\cos\tau + a_1\sin\tau) - \delta_4\lambda_1(c_1\cos\tau - d_1\sin\tau),$$

$$\ddot{\beta}_1 - 2\dot{\gamma}_1 - \delta_3\lambda_0'\xi_1 + \delta_4\lambda_0'\beta_1 = (d_1\cos\tau + c_1\sin\tau)\sin^2\tau +$$

$$+(c_1\cos\tau - d_1\sin\tau)\frac{\sin 2\tau}{2} + (d_1\sin\tau - c_1\cos\tau)\sin 2\tau +$$

$$+(c_1\sin\tau + d_1\cos\tau)\cos 2\tau + \delta_3\lambda_1(a_1\cos\tau - b_1\sin\tau) - \delta_4\lambda_1(d_1\cos\tau + c_1\sin\tau).$$

Eliminating the secular terms, we find that a) for $c_1 = 0,\ d_1 \neq 0$,

$$\lambda_1 = \frac{\delta_1}{\delta_1\delta_4 - \delta_2\delta_3};$$

b) for $d_1 = 0,\ c_1 \neq 0$,

$$\lambda_1 = 0.$$

Thus, near the point λ_0' there exists a region of instability determined by the inequalities

$$\lambda_0' \leq \lambda \leq \lambda_0' + \frac{\delta_1\varepsilon}{\delta_1\delta_4 - \delta_2\delta_3}.$$

Putting $n = 1$, $\omega = \sqrt{1/\lambda_0''}$, and carrying out analogous operations, we find that the instability region degenerates in the first approximation into the straight line

$$\lambda = \lambda_0'' + \frac{\delta_1\varepsilon}{2(\delta_1\delta_4 - \delta_2\delta_3)}.$$

Differential equations (7) and (8) make it possible to determine the boundaries of other regions of instability.

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Note: Figure translations are in progress. See original paper for figures.