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Abstract

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ON A FINITE-DIFFERENCE METHOD FOR SOLVING THE ONE-PHASE STEFAN PROBLEM FOR A QUASILINEAR EQUATION

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1°. Let a heat flux $q_2(t)$ be supplied to the right end $x = l$ of the rod $0 \leq x \leq l$, with a thermally insulated lateral surface, and let a heat flux $q_1(t)$ be removed from the left end, where $0 \leq q_1(t) < q_2(t)$. Suppose that at $t = 0$ the rod is uniformly heated to a temperature below the melting temperature and suppose that $q_2(t) - q_1(t) \geq \text{const} > 0$. Then, for some $t = t^*$, the right end of the rod will begin to melt. Finding the temperature distribution in the rod for $0 \leq t \leq t^*$ is the well-known classical heat-conduction problem, for whose solution there exist many different methods. Therefore we shall assume the temperature distribution in the rod at the instant $t = t^*$ to be known. Shifting the time origin to the point $t = t^*$ and taking the melting temperature of the rod to be zero, the problem of heating and melting the rod with removal of the molten material can be formulated as follows: it is required to find the temperature $u(x, t)$ of the rod and the length $x = y(t)$ of the rod from the conditions:

$$a(u)u_{xx} - u_t = 0 \quad \text{in } D = \{0 < x < y(t),\ 0 < t < T\}; \tag{1}$$

$$u_x(0, t) = q_1(t), \quad 0 < t < T; \qquad u(y(t), t) = 0, \quad 0 \leq t \leq T; \tag{2}$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l, \qquad \varphi(l) = 0; \tag{3}$$

$$\gamma y'(t) = u_x(y(t), t) - q_2(t), \quad 0 < t < T, \qquad y(0) = l, \tag{4}$$

where $\gamma = \text{const} > 0$; $a(u)$ are the physical characteristics of the rod; T is the time of complete melting of the rod, i.e. $y(T) = 0$. An analogous problem was considered in a quasistationary formulation in ([1]). By reduction to integral equations, in ([2]) the "local" problem with more general boundary conditions is studied for a quasilinear equation with linear principal part $a^2 u_{xx} - u_t$, where $a = \text{const}$.

In the present note the existence and uniqueness of the solution of problem (1)–(4) in the sense indicated below are proved under certain restrictions on the parameters of the problem; for finding an approximate solution of the stated problem a certain effective implicit difference scheme is proposed (Theorem 1), and the convergence of the approximate solution obtained by this scheme to the solution of problem (1)–(4) is justified (Theorem 2). In doing this we partly used the methodology of paper ([3]).

Definition. The functions $u(x,t), y(t)$ will be called a **solution of problem** (1)–(4) if: 1) $y(t)$ is continuously differentiable for $0 \le t \le T$; $y(t) > 0$ for $0 \le t < T$, $y(T) = 0$, $y'(t) < 0$; 2) $u(x,t)$ is defined and continuous in $\overline{D} = \{0 \le x \le y(t),\ 0 \le t \le T\}$ together with the derivative $u_x(x,t)$, and, moreover, has continuous in D and bounded in $\overline{D}$ derivatives $u_{xx}(x,t), u_t(x,t)$; 3) all conditions (1)–(4) are satisfied.

2°. Divide the interval $0 \le x \le l$ by the points x_i into N equal parts with step h, where $x_i = ih,\ i = 0,1,2,\dots,N,\ x_N = l$. The time steps τ_n will be chosen depending on n in such a way that for each $t = t_n = \sum_{k=1}^{n} \tau_k$

the broken line approximating the curve $y(t)$ should fall on the node with coordinates $(y_n = (N-n)h, t_n)$, shifting at each time step by the amount h, i.e., $y_n - y_{n-1} = -h$, where y_n is the approximate value of the length of the rod at the moment $t = t_n$. We replace problem (1)–(4) by the following difference problem for determining τ_n and the approximate values w_{in} of the temperature $u(x_i,t_n)$:

$$a(w_{i,n-1})\,\delta_{xx} w_{in} - \delta_t^- w_{in} = 0, \qquad 1 \le i \le N_n - 1, \quad 1 \le n \le N-2; \tag{5}$$

$$\delta_x w_{0,n} = q_{1,n-1}, \qquad 1 \le n \le N-1; \qquad w_{N_n,n} = 0, \qquad 1 \le n \le N; \tag{6}$$

$$w_{i0} = \varphi_i, \qquad 0 \le i \le N, \tag{7}$$

$$-\gamma\,\frac{h}{\tau_n} = \delta_x w_{N_n-1,n} - q_{2,n-1}, \qquad 1 \le n \le N-1; \qquad y_0 = l, \tag{8}$$

where

$$\delta_x w_{in} = \frac{1}{h}(w_{i+1,n} - w_{in}), \qquad \delta_{xx} w_{in} = \frac{1}{h^2}(w_{i+1,n} - 2w_{in} + w_{i-1,n}),$$

$$\delta_t^- w_{in} = \frac{1}{\tau_n}(w_{in} - w_{i,n-1}), \qquad y_n = N_{nh}, \qquad N_n = N - n,$$

$$t_n = \sum_{k=1}^{n} \tau_k, \qquad q_{i,n} = q_i(t_n), \qquad \varphi_i = \varphi(x_i), \qquad \tau_N = \tau_{N-1}.$$

The system (5)–(8) is nonlinear with respect to the unknowns w_{in}, τ_n. Assuming that $w_{i,k}, \tau_k,\ 1 \le k \le n-1$, satisfying (5)–(8), are known, to determine $w_{i,n}, \tau_n$ $(n = 1, 2, ...)$ we apply the iteration method according to the scheme:

$$a(w_{i,n-1})\, \delta_{xx} w_{i,n}^{(s)} - \delta_t^- w_{i,n}^{(s)} = 0, \qquad 1 \le i \le N_n - 1; \tag{9}$$

$$\delta_t^- w_{i,n}^{(s)} = \frac{1}{\tau_n^{(s)}} (w_{in}^{(s)} - w_{i,n-1});$$

$$\delta_x w_{0,n}^{(s)} = q_{1,n-1}, \qquad w_{N_n,n}^{(s)} = 0, \qquad w_{i,0}^{(s)} = \varphi_i; \tag{10}$$

$$\tau_n^{(s+1)} = \frac{1}{q_{2,n-1} - q_{1,n-1}} \left[\gamma h + \tau_n^{(s)} \left(\delta_x w_{N_n-1,n}^{(s)} - q_{1,n-1}\right)\right], \tag{11}$$

where $\tau_n^{(0)} > 0,\ s = 0, 1, 2, ...$ Knowing $\tau_n^{(0)} > 0$, from (9), (10) for $s = 0$ we find $w_{i,n}^{(0)}$, and from (11) for $s = 0$ we then obtain $\tau_n^{(1)}$, and so on. In solving system (9), (10), one may use, for example, the sweep method [(4)].

Theorem 1. Suppose $q_1(t'') \le q_1(t')$ for any $t'' \ge t' \ge 0$, $q_2(t) > q_1(t) \ge 0$; $\varphi(l) = 0$, $\delta_x \varphi_i \ge q_1(0)$, $\delta_{xx} \varphi_i \ge 0$ for $0 \le x_i \le l$; $a(u) \ge a_0 > 0$ for $\varphi_0 \le u \le 0$.

Then, for any prescribed $\tau_n^{(0)} > 0$, the iterations will be uniquely determined for every $s \ge 0$ and, as $s \to \infty$, $w_{i,n}^{(s)}, \tau_n^{(s)}$, varying monotonically, will converge to the solution $w_{i,n}, \tau_n$ of system (5)–(8). Moreover, the following relations hold:

$$q_{1,n-1} \le \delta_x w_{in} \le q_{2,n-1}, \qquad \delta_{xx} w_{i,n} \ge 0, \qquad \delta_t^- w_{i,n} \ge 0, \qquad \varphi_0 \le w_{in} \le 0, \tag{12}$$

$$0 \le i \le N_n, \qquad n = 0, 1, ...;$$

$$0 < \frac{h}{\tau_n} = -\frac{y_n - y_{n-1}}{t_n - t_{n-1}} \le \frac{q_2 - q_1}{\gamma} = \Lambda_1; \tag{13}$$

$$t_n = l + \frac{1}{\gamma}\left[h \sum_{i=1}^{N-n-1} F_{i,n} + h \sum_{i=N-n}^{N-2} F_{i,N-i-1} - h \sum_{i=1}^{N-2} F_{i,0} - \sum_{m=1}^{n} \tau_m (q_{2,m-1} - q_{1,m-1})\right]; \tag{14}$$

where

$$F_{i,m} = -\sum_{k=m+1}^{N-i} \frac{w_{i,k} - w_{i,k-1}}{a(w_{i,k-1})}, \qquad q_i = \min_{t \geq 0} q_i(t), \qquad Q_i = \max_{t \geq 0} q_i(t).$$

For the proof we note that $p_{in} = \delta_t w_{in}^{(s)}$ satisfies the conditions

$$p_{in} = \tau_n a(w_{i,n-1}) \delta_{xx} p_{i,n} + a(w_{i,n-1}) \delta_{xx} w_{i,n-1}, \qquad 1 \leq i \leq N_n - 1;$$

$$\delta_x p_{0n} = \delta_t q_{1,n-1} \leq 0; \qquad p_{N_n,n} = \frac{h}{\tau_n} \delta_x w_{N_n,n-1}.$$

Hence, by means of the maximum principle and induction with respect to n, we obtain the inequalities:

$$\delta_x w_{in}^{(s)} \geq q_{1,n-1}, \qquad \delta_{xx} w_{in}^{(s)} \geq 0, \qquad \delta_t w_{in}^{(s)} \geq 0, \qquad \varphi_0 \leq w_{in}^{(s)} \leq 0,$$

$$\tau_n^{(s+1)} \geq \frac{\gamma h}{q_{2,n-1} - q_{1,n-1}} > 0$$

for any prescribed $\tau_n^{(0)} > 0$ for all $s \geq 0$. From these inequalities, and also from the maximum principle for the system

$$z_{in}^{(s)} - a(w_{i,n-1}) \tau_n^{(s)} \delta_{xx} z_{in}^{(s)} + a(w_{i,n-1}) [\tau_n^{(s)} - \tau_n^{(s-1)}] \delta_{xx} w_{i,n}^{(s-1)},$$

$$\delta_x z_{0,n}^{(s)} = 0, \qquad z_{N_n,n}^{(s)} = 0, \qquad \tau_n^{(s+1)} - \tau_n^{(s)} = \frac{h}{q_{2,n-1} - q_{1,n-1}} \sum_{i=1}^{N_n - 1} \frac{z_{i,n}^{(s)}}{a(w_{i,n-1})},$$

where $z_{i,n}^{(s)} = w_{i,n}^{(s)} - w_{i,n}^{(s-1)}$, there follows the monotone convergence of $w_{i,n}^{(s)}$, τ_n^s, as $s \to \infty$, to the solution w_{in}, τ_n of the system (5)–(8). In this case the direction of monotonicity depends on $\tau_n^{(0)} > 0$; in particular, if

$$0 < \tau_n^{(0)} \leq \frac{\gamma h}{q_{2,n-1} - q_{1,n-1}},$$

then $w_{i,n}^{(s)}$, $\tau_n^{(s)}$ increase monotonically as s grows, while if $\tau_n^{(0)} > 0$ is a sufficiently large number, they decrease monotonically. The estimate $\delta_x w_{in} \leq q_{2,n-1}$ follows from (8).

3°. We join the points (y_n, t_n) by line segments and denote the polygonal line obtained by $y(t,h)$, setting $y(t,h) \equiv 0$ for $t \geq t_N$. We extend the mesh function w_{in} to the whole region $D_h = \{0 \leq x \leq y(t,h),\ 0 \leq t \leq t_N\}$ continuously, as is done in (5), p. 359; the function obtained is denoted by $u(x,t,h)$, setting $u(x,t,h) \equiv 0$ for $x \geq y(t,h)$, $t \geq 0$.

Theorem 2. *Suppose the following conditions are satisfied:*

1) *$\varphi(x)$ on the interval $0 \leq x \leq l$ has a 4-times continuous derivative, $\varphi(l) = 0$,*

$$\varphi'(0) = -q_1(0), \qquad \varphi''(0) = \varphi'''(0) = 0, \qquad \delta_x \varphi_i \geq q_1, \quad 0 \leq a(\varphi(x_i))\delta_{xx}\varphi_i \leq M_2;$$

2) *$q_1(t) \equiv q_1 = \mathrm{const} \geq 0$, $q_2(t)$ is continuous and bounded for $t \geq 0$, and*

$$a_0(q_2 - q_1) > lM_2; \qquad \frac{1}{\gamma}(Q_2 - q_1) < \frac{a_0}{Q_2 l}(q_2 - q_1); \tag{15}$$

3) *$a(u)$ on the interval $\varphi_0 - \alpha \leq u \leq 0$, where $\alpha > 0$ is an arbitrarily small but fixed number, has 6 continuous derivatives, and $a'(u) \leq 0$, $a(u) \geq a_0 > 0$.*

Then the problem (1)–(4) has a unique solution $u(x,t), y(t)$, which can be obtained as the limit, as $h \to 0$, of the functions $u(x,t,h), y(t,h)$.

Proof. From (5)–(8), (12), (13), (15) it follows that

$$\delta_t w_{i,n} \leq \mu_2 = \max\{M_2, Q_2\Lambda_1\}, \qquad 0 < \Lambda_2 \leq \frac{h}{\tau_n} \leq \Lambda_1, \tag{16}$$

where

$$\Lambda_2 = \frac{a_0(q_2 - q_1) - \mu_2 l}{\gamma a_0},$$

whence

$$\max_{1 \leq n \leq N} \tau_n \to 0$$

as $h \to 0$. In view of the inequalities (16), $0 \leq y(t,h) \leq l$, and Arzelà' s theorem, there exists a subsequen-

there exists a sequence h_ν such that $\lim_{\nu\to\infty} h_\nu = 0$, and $y(t, h_\nu)$ converges uniformly for $t \geqslant 0$ to the curve $y(t)$, and

$$\begin{aligned} &y(t) > 0 \quad \text{for } 0 \leqslant t < T, \qquad y(T) = 0, \qquad y(0) = l, \\ &-\Lambda_1 \leqslant \frac{y(t'') - y(t')}{t'' - t'} \leqslant -\Lambda_2 < 0, \qquad t'', t' \in [0, T]. \end{aligned} \tag{17}$$

Lemma. If $q_1(t), \varphi(x), a(u)$ satisfy all the conditions of Theorem 2, then the boundary-value problem (1)–(3) for a prescribed curve $y(t)$ with properties (17) has a unique solution $u(x,t)$, continuous in $\overline{D}\{0 \leqslant x \leqslant y(t),\, 0 \leqslant t \leqslant T\}$ together with the derivative $u_x(x,t)$, and possessing continuous derivatives u_{tt}, u_{xxxx} in $D\{0 < x < y(t),\, 0 < t < T\}$, where

$$q_1 \leqslant u_x \leqslant Q_2, \qquad 0 \leqslant u_{xx} \leqslant \frac{\mu_2}{a_0} \quad \text{in } \overline{D}, \qquad |u_{xxxx}| \leqslant M_4(\delta), \qquad |u_{tt}| \leqslant M_2(\delta) \tag{18}$$

in subdomains of the form $D_\delta\{0 \leqslant x \leqslant y(t) - \delta,\, 0 \leqslant t \leqslant T_\delta\}$ for any $\delta > 0$; $y(T_\delta) = \delta < l$.

The proof of this lemma can be obtained by the finite-difference method from (5)–(7) with the aid of inequalities (18) and the estimates of S. N. Bernstein ([5],[6]). Put $r(x,t) = u(x,t) - u(x,t,h_\nu)$. By virtue of (12), (18) and the uniform convergence of $y(t,h_\nu)$ to $y(t)$, for all sufficiently large ν outside D_δ the estimate $|r(x,t)| \leqslant 2Q_2\delta$ holds. Inside D_δ, $r_{i,n} = r(x_i,t_n)$ satisfies the conditions:

$$a(w_{i,n-1})\, \delta^-_{xx} r_{i,n} - \delta^-_t r_{i,n} = R_{i,n}(h_\nu), \qquad \delta_x r_{0,n} = \frac{1}{2} h_\nu u_{xx}(\overline{x},t),$$

$$r_{i,0} = 0, \qquad |r_{i,n}|_\Gamma \leqslant 2Q_2\delta,$$

where, by (16) and (18), $|R_{i,n}(h_\nu)| \leqslant M(\delta)h_\nu$, $0 < \overline{x} < h_\nu$; Γ is the polygonal line consisting of segments joining the grid nodes and lying between $y(t) - \delta$ and $y(t) - \frac{3}{4}\delta$. Hence, using (12), (18), we obtain the uniform convergence of $u(x,t,h_\nu)$ to $u(x,t)$ in $\overline{D}$ as $\nu \to \infty$. Passing to the limit in (14), we have

$$y(t) = l + \frac{1}{\gamma}\left\{\int_0^{y(t)} F(u(x,t))\,dx - \int_0^l F(\varphi(x))\,dx - \int_0^t [q_2(\tau) - q_1(\tau)]\,d\tau\right\}, \tag{19}$$

where

$$F(u) = \int_0^u \frac{ds}{a(s)}.$$

It follows from this that $y(t)$ has a continuous derivative and that condition (4) is satisfied.

Since every solution of problem (1)–(4) satisfies identity (19), using the theorems of P. Vyborny and L. Nirenberg ([6]), it is not difficult to prove uniqueness of the solution, following the scheme of ([7]). From the uniqueness of the solution of

(1)–(4) it follows that $u(x, t, h), y(t, h)$ tend to the solution of (1)–(4) for any manner in which h tends to zero.

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Note: Figure translations are in progress. See original paper for figures.