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Abstract

Full Text

MATHEMATICS

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CONSTRUCTION OF LOCAL AND GLOBAL HOMEOMORPHISMS FOR ONE CLASS OF EQUATIONS IN QUATERNIONS

(Presented by Academician I. N. Vekua, June 13, 1963)

It is known that in the plane case ($n = 2$) the Beltrami equation $W_{\bar{z}} - q(z)W_z = 0$ always has homeomorphic solutions (for more detail see ([1])). Of interest is the question of the existence of homeomorphic solutions (homeomorphisms) for the spatial Cauchy–Riemann system ($n \geqslant 3$) and for elliptic systems related to it.

It is known that the number of components of a holomorphic vector (i.e., a solution of the Cauchy–Riemann system) coincides with the dimension of the space (and with the number of equations) only in the cases $n = 2, 4, 8$. Since under homeomorphisms the dimension of the space remains invariant, the Cauchy–Riemann system admits solutions carrying out topological mappings only in two-, four-, and eight-dimensional spaces. In particular, in the three-dimensional case the Cauchy–Riemann system, generally speaking, has no homeomorphic solutions (the holomorphic vector has 4 components). However, as shown in ([4]), for an elliptic system related to the Cauchy–Riemann conditions there always exists a solution, any three components of which carry out a local homeomorphism of the space E_3 onto the space determined by these components.

In the present note we show the existence of a global homeomorphism for an elliptic system of special form in four-dimensional space. For the proof we make extensive use of the ideas of the book ([1]). The eight-dimensional case can be considered analogously.

Consider the system

$$\sum_{k=1}^{4} A_k(x) \frac{\partial}{\partial x_k} U = 0, \tag{1}$$

where $U(x)$ is an unknown four-component real vector, $A_k(x)$, $k = 1, 2, 3, 4$, are matrices of quaternion type, i.e.,

$$A_k(x) = a_{k1}(x)e + a_{k2}(x)i + a_{k3}(x)j + a_{k4}(x)k,$$

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where the matrices i, j, k have the form

$$i = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \quad j = \begin{Vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, \quad k = \begin{Vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix},$$

e is the identity matrix of order four. The $a_{kl}(x)$ are real functions of the point $x(x_1, x_2, x_3, x_4)$ of a finite domain G of four-dimensional Euclidean space E_4, bounded by a twice continuously differentiable surface.

If system (1) is elliptic, then it can always be written in the form (by changing, perhaps, the independent variables)

$$D_1U - Q_2D_2U - Q_3D_3U - Q_4D_4U = 0, \tag{2}$$

where the matrices $Q_l(x)$, $l = 1, 2, 3, 4$, are easily expressed in terms of $A_k(x)$, $k = 1, 2, 3, 4$, and the operators D_l have the form:

$$D_1 = e\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} + j\frac{\partial}{\partial x_3} + k\frac{\partial}{\partial x_4},$$

$$D_2 = e\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - j\frac{\partial}{\partial x_3} - k\frac{\partial}{\partial x_4},$$

$$D_3 = e\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} + j\frac{\partial}{\partial x_3} - k\frac{\partial}{\partial x_4},$$

$$D_4 = e\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} - j\frac{\partial}{\partial x_3} + k\frac{\partial}{\partial x_4}.$$

For simplicity we shall restrict ourselves to considering the system

$$D_1U - QD_2U = 0, \tag{3}$$

where

$$Q(x) = eq_1(x) + iq_2(x) + jq_3(x) + kq_4(x),$$

although all the results are valid for the more general system (2).

Suppose that system (3) is elliptic in the sense of Petrovsky. This means (see ([4])) that everywhere in the domain G

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$$(1-r^2)^2 + 4q_2^2 \neq 0, \qquad r^2(x) = \sum_{i=1}^{4} q_i^2(x).$$

It is natural to assume that the stronger inequality is satisfied

$$r(x) \leq q_0 < 1, \qquad q_0 = \text{const.} \tag{4}$$

Inequality (4) makes it possible to construct, for system (3), generalized solutions of the class $W_2^{(1)}$, assuming only the measurability of the functions $q_i(x)$. The general scheme for constructing homeomorphic solutions is as follows.

Using the idea of I. N. Vekua (see ([1])), we seek a solution of system (3) in the form (cf. ([4]))

$$\mathcal{W}(x) = Z + T\omega, \tag{5}$$

where

$$Z = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix}, \qquad T\omega = -\frac{1}{4\pi^2} \iint_G \overline{D_1} \frac{1}{|x-\xi|^2}\, \omega(\xi)\, d\xi,$$

which gives, for the four-component vector ω, the singular integral equation

$$\omega - Q\Pi_2\omega = Q_1, \tag{6}$$

where $\Pi_2\omega \equiv D_2 T\omega$ and $Q_1 = QD_2Z$.

To prove the solvability of equation (6), the contraction mapping principle is used in one of the functional spaces L_2, L_p, $W_p^{(1)}$, $p > 4$, or C_α.

Theorem 1. Let $Q(x) \in C_\alpha(\overline{G})$. If system (3) is elliptic in the sense of Petrovsky, then in some sufficiently small neighborhood G_0 of any point x_0 of the domain G there exists a solution $\mathcal{W}_0(x)$ of system (3), which realizes a local homeomorphism of the space E_4 onto the space W_4 defined by the components of the solution. Moreover, $\mathcal{W}_0(x) \in C_\alpha^1(\overline{G}_0)$.

Analogously ([1]), theorems can be proved concerning the smoothness of the constructed solution $\mathcal{W}_0(x)$ depending on the smoothness of the matrix $Q(x)$.

Theorem 2. If $Q(x) \in W_p^{(1)}(\overline{G})$, $p > 4$, and the norm of the matrix $Q(x)$ in this space is sufficiently small, i.e. the quantities $\|q_i(x)\|_{W_p^{(1)}}$, $i = 1, 2, 3, 4$, are small, then there always exists a global homeomorphism $\mathcal{W}(x)$ of system (3),

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i.e. there exists a four-component vector function $\mathcal{W}(x)$, satisfying system (3) everywhere in $\overline{G}$ and mapping the domain G onto some domain G_W of the four-dimensional space W_4. This mapping is ε-quasiconformal and belongs to $C^1_\alpha(\overline{G})$, where $\alpha = \dfrac{p-4}{p}$. If, moreover, $Q(x) \in W^{(1)}_p(E_4)$, $p > 4$, with $Q(x) \equiv 0$ outside some sufficiently large ball, and the norm of the matrix $Q(x)$ in this space is still sufficiently small, then there exists a complete homeomorphism $\mathcal{W}_*(x)$ of system (3) from the space E_4 onto the space W_4. In this case

$$\mathcal{W}_*(x) - Z \in C^1_\alpha(E_4), \qquad \alpha = \frac{p-4}{p},$$

and $\mathcal{W}_*(x)$ realizes an ε-quasiconformal mapping of the space E_4 onto the space W_4.

For the proof, the matrix $Q(x)$ is extended, with preservation of its class, to the whole space, in such a way that $Q(x) \equiv 0$ in a neighborhood of infinity, and a solution $\mathcal{W}(x)$ of the form (5) is constructed whose Jacobian is everywhere different from zero. By virtue of the conditions on $Q(x)$, it possesses sufficient smoothness. To show the one-to-one character of the constructed solution, we use the Lefschetz-Hopf theorem on the algebraic number of fixed points of a vector field (see ([3])); hence the mapping $\mathcal{W}(x)$ is topological and therefore, by Brouwer' s theorem (see ([3])), maps the domain G onto some domain G_W of the space W_4.

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Note: Figure translations are in progress. See original paper for figures.

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