MATHEMATICS

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Abstract

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ON BASES IN SPACES OF CONTINUOUS FUNCTIONS

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§ 1. Let R and S be sets in a Banach space E; let P and Q be the closures of their linear spans. We shall call the inclination of R to S the quantity

$$(\widehat{R;S}) = \inf_{x\in P;\ \|x\|=1} \rho(x;Q).$$

For a sequence $\{e_i\}_{i=1}^{\infty}$, $e_i \in E$, $i = 1,2,\ldots$, denote by P_{ij} the linear span of the elements $e_i, e_{i+1}, \ldots, e_j$ $(i \leq j)$, and we shall call the index of $\{e_i\}_{i=1}^{\infty}$ the quantity

$$\gamma_{\{e_i\}} = \inf_{n;\ m;\ n<m} (\widehat{P_{1;n}; P_{n+1;m}}).$$

Obviously, $0 \leq \gamma_{\{e_i\}} \leq 1$. If $\gamma_{\{e_i\}} = 1$, then $\{e_i\}_{i=1}^{\infty}$ is called orthogonal*. As M. M. Grinblum showed (1), in order that the sequence $\{e_i\}_{i=1}^{\infty}$ be a basis** in E, it is necessary and sufficient that $\gamma_{\{e_i\}} > 0$.

We adopt the notation: C is the space of all functions continuous on $[0,1]$ with norm $\|\varphi\| = \max_{0\leq t\leq 1} |\varphi(t)|$, and $\widetilde{C}$ is the space of all real-valued functions continuous on $[0,1]$ with the same norm.

Theorem 1. *If a sequence of functions differentiable on* $(0,1)$, $\{e_i(t)\}_{i=1}^{\infty}$, *in* $\widetilde{C}$ *satisfies the conditions:*

1) *the set of stationary points of any linear combination*

$$\sum_{i=1}^{n} \alpha_i e_i(t)$$

has no limit points in $(0,1)$*;*

2) *in the closure of the linear span of* $\{e_i\}_{i=1}^{\infty}$ *there exists a function which, together with its derivative, preserves its sign on* $(0,1)$*, then* $\gamma_{\{e_i\}} < 1$.

Denote by M_f the set of points of greatest value on $[0,1]$ of the function $|f(t)|$.

Lemma 1. *For a given function $f(t) \in \widetilde{C}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any function $g(t) \in \widetilde{C}$, $\|g\| < \delta$:*

$$M_{f+g} \subset \bigcup_{t \in M_f} (t-\varepsilon,\ t+\varepsilon).$$

* In works by foreign authors such a sequence is sometimes called monotone, apparently proceeding from the fact that

$$\left\|\sum_{i=1}^{m} \alpha_i e_i\right\| \geq \left\|\sum_{i=1}^{n} \alpha_i e_i\right\| \quad \text{for } m \geq n.$$

** The sequence $\{e_i\}_{i=1}^{\infty}$ is called a basis in E if it is a basis in the closure of the linear span over it.

Lemma 2. Let t', t'' be respectively the least and the greatest of the values of t at which the maximum value of the modulus of the function $f(t)$, differentiable on $(0,1)$, is attained, $0 < t' < t'' < 1$, and for some function $g(t)$ let, on the interval $(a,b) \supset M_f$, $g(t) > 0$ and $g'(t)$ preserve its sign. Then there exists $\alpha_0 > 0$ such that, for $0 < \alpha < \alpha_0$,

$$M_{f+\alpha g} \subset (t'', b), \quad \text{if } g'(t) > 0 \text{ and } f(t'') > 0,$$

$$M_{f+\alpha g} \subset (a, t'), \quad \text{if } g'(t) < 0 \text{ and } f(t') > 0.$$

Lemma 3. In order that $(f;g) < 1$, $f \in \widetilde{C}$, $g \in \widetilde{C}$, it is necessary and sufficient that one of the following conditions be fulfilled:

$$\begin{aligned} &1) \quad \operatorname{sign} f(t) = \operatorname{sign} g(t) \quad \text{for } t \in M_f, \\ &2) \quad \operatorname{sign} f(t) = -\operatorname{sign} g(t) \quad \text{for } t \in M_f. \end{aligned}$$

Let us outline the proof of Theorem 1. There is a function $f_1(t) \in P_{1;3}$, $f_1(0) = f_1(1) = 0$, and if t_0 is the greatest of the values of t for which $|f_1(t)|$ has its maximum value on $[0,1]$, then $0 < t_0 < 1$. For definiteness assume that $f_1(t_0) > 0$, and for the function $g(t)$ defined in condition 2 of the theorem we have $g(t) > 0$, $g'(t) > 0$, $0 < t < 1$.

It follows from condition 1 that on some interval (t_0, τ), $f_1'(t) < 0$. From Lemmas 1 and 2 it follows that for some natural number n there is a function $f_2(t) \in P_{1;n}$ such that $M_{f_2} \subset (t_0, \tau)$ and $f_2(t_0') > 0$, where t_0' is the least of the points of maximum value of $|f_2(t)|$ on $[0,1]$; $t_0 < t_0' < \tau$. Consider any function

$h(t) \in P_{1;N}$, where N is arbitrary. On the basis of condition 1 and Rolle' s theorem, $h(t)$ preserves its sign on some interval (τ_0, t_0'), $t_0 < \tau_0 < t_0'$. By Lemma 2 we obtain that, for some $\alpha > 0$, the function $f_3(t) = f_2(t) + \alpha f_1(t)$ satisfies the conditions:

$$M_{f_3} \subset (\tau_0, t_0') \quad \text{and} \quad f_3(t) > 0 \text{ for } t \in M_{f_3}.$$

But then, by Lemma 3, $(f_3; h) < 1$, and, since $f_3(t) \in P_{1;n}$, it follows that $(P_{1;n}; h) < 1$. Hence $\gamma_{\{e_i\}} < 1$.

Corollary. In the space $\widetilde{C}$ there does not exist an orthogonal basis consisting of functions analytic on $(0, 1)$.

This, in particular, means the impossibility of constructing in $\widetilde{C}$ an orthogonal basis consisting of polynomials or trigonometric polynomials.

We note that in $\widetilde{C}$ there exist bases of polynomials that are orthogonal in the metric L^2 ([2]).

With the aid of Theorem 1 one establishes

Theorem 2. If the elements of the space $E \subset \widetilde{C}$ are functions analytic on $(0, 1)$ and, moreover, there exists a function $g(t) \in E$ which preserves its sign on $(0, 1)$ together with its first derivative, then there is no orthogonal basis in E.

Examples of spaces satisfying the conditions of Theorem 2 may be the closures in $\widetilde{C}$ of the linear spans of the powers $\{t^{n_k}\}_{k=1}^{\infty}$,

$$n_k > 0, \quad \sum_{k=1}^{\infty} \frac{1}{n_k} < \infty \quad ({}^3).$$

Thus the question of the existence of infinite-dimensional separable Banach spaces not having an orthogonal basis is answered.

§ 2. A subspace $E \subset C$ will be called **saturated** if, whatever the partition

$$[0, 1) = \bigcup_{i=0}^{n-1} [a_i, a_{i+1}), \qquad 0 = a_0 < a_1 < \cdots$$

$\cdots < a_n = 1$, and the function $f(t) \in C$, there exist functions $g(t) \in E$ and a collection of points $\{t_i\}_{i=1}^{n}$, $t_i \in (a_{i-1} a_i)$, $i = 1, 2, ...$, such that $f(t_i) = g(t_i)$. With the aid of the Banach–Mazur theorem ([4]) one can obtain

Theorem 3. *Every infinite-dimensional separable Banach space is isomorphic to a saturated subspace of the space C.*

Definition. Let $\mathfrak{M}$ be some class of subsequences of the sequence of natural numbers; a sequence $\{e_i\}_{i=1}^{\infty}$ will be called a **basis in E relative to $\mathfrak{M}$** if it is complete in E and every subsequence $\{e_{n_k}\}_{k=1}^{\infty}$ for which $\{n_k\}_{k=1}^{\infty} \in \mathfrak{M}$ is a basis in E.

Example 1. The sequence $\{s_i(t)\}_{i=1}^{\infty}$, where $s_{2k-1} = \cos(k-1)t$, $s_{2k} = \sin kt$, $k = 1, 2, ...$, is a basis in C relative to the class of all lacunary subsequences of the sequence of natural numbers ([5]).

Example 2. The sequence of powers $\{t^k\}_{k=1}^{\infty}$ is a basis in C relative to the class of all subsequences of the sequence of natural numbers satisfying the condition:

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} > A,$$

where A is an absolute constant.

The last assertion is a special case of the following theorem.

Theorem 4. *In order that the sequence of powers $\{t^{n_k}\}_{k=1}^{\infty}$, where $\{n_k\}_{k=1}^{\infty}$ is a positive increasing sequence, be a basis in C, it is necessary that*

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} > 1,$$

and it is sufficient that

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} > A,$$

where A is an absolute constant.

Let $\{p_k\}_{k=1}^{\infty}$ be a subsequence of the sequence of natural numbers. Denote by $\mathfrak{M}_{\{p_k\}}$ the class of all subsequences of the sequence of natural numbers each of which does not contain at least one number from each pair of the form $\{p_k,\ p_k + 1\}$, $k = 1, 2, ...$

Theorem 5. *In an infinite-dimensional separable Banach space, for any subsequence $\{p_i\}_{i=1}^{\infty}$ of the sequence of natural numbers there exists a basis relative to $\mathfrak{M}_{\{p_i\}}$.*

The idea of the proof consists in constructing a basis relative to $\mathfrak{M}_{\{p_i\}}$ in an arbitrary saturated subspace of the space C, followed by an application of Theorem 3.

Putting $p_i = i$, $i = 1, 2, ...$, in Theorem 5, we obtain

Corollary. In an infinite-dimensional separable Banach space E there exists a complete sequence $\{e_i\}_{i=1}^{\infty}$ such that any subsequence $\{e_{n_i}\}_{i=1}^{\infty}$ for which $n_{i+1} - n_i > 1$, $i = 1, 2, ...$, is a basis in E.

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Note: Figure translations are in progress. See original paper for figures.