O. A. OLEINIK

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Abstract

Full Text

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ON THE SYSTEM OF PRANDTL EQUATIONS IN THE THEORY OF THE BOUNDARY LAYER

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In this paper we study the system of boundary-layer equations for an incompressible fluid

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}-\frac{dp}{dx},\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \tag{1}$$

in the domain $D_A\{0<x<A, 0<y<\infty\}$, with conditions

$$u|_{y=0}=0,\qquad v|_{y=0}=v_0(x),\qquad u|_{x=0}=u_0(y),\qquad u(x,y)\to U(x)\quad \text{as } y\to\infty \tag{2}$$

uniformly in x, where $U^2(x)+2p(x)=C$ (Bernoulli' s law). Under natural assumptions concerning $v_0(x)$, $u_0(y)$, $p(x)$, a solution of the problem (1), (2) in the domain D_A has been obtained for some $A>0$, and its uniqueness has been proved. As is known (see (^{1}), Ch. IV), a solution of the system (1) with conditions (2) may fail to exist in D_A for large A, as a consequence of separation of the boundary layer.

We shall assume that $u_0(y)>0$ for $y>0$; $u_0(0)=0$; $u_0'(0)>0$; $u_0(y)\to U(0)\neq 0$ as $y\to\infty$; dp/dx and $v_0(x)$ are continuously differentiable on $[0,A]$; u_0, u_0', u_0'' are bounded for $0\leq y<\infty$ and satisfy the Hölder condition. We also assume that the compatibility condition at the point $(0,0)$ is fulfilled:

$$\nu u_0''(y)-p'(0)-v_0(0)u_0'(y)=O(y^2) \tag{3}$$

for small y. Let $\bar{D}_A$ be the closure of D_A.

1. Existence theorem. *In the domain D_A, for some $A>0$, there exists a solution u,v of the system (1) with conditions (2) such that u is continuous and bounded in $\bar{D}_A$, $u>0$ for $y>0$, $k_1y\geq u\geq k_2y$ for $0<y\leq y_0$, where y_0, $k_1>0$, $k_2>0$ are certain constants; $\partial u/\partial y$ and $\partial^2u/\partial y^2$ are continuous and bounded in $\bar{D}_A$; $\partial u/\partial x$, v, $\partial v/\partial y$ are continuous and bounded in any finite part of the domain $\bar{D}_A$. If $u_0'(y)\to 0$ as $y\to\infty$ sufficiently rapidly, then $\partial u/\partial x$,*

$\partial v/\partial y$ are bounded in D_A. The value of A depends on u_0, v_0, p, U. In the case $dp/dx \le 0$, $v_0(x) \le 0$, such a solution of the problem (1), (2) exists in D_A for any $A < \infty$.

The proof of this theorem is based on the following proposition.

Lemma. *For the existence of a solution u, v of the problem (1), (2) possessing the properties indicated in the theorem, it is sufficient that there exist, in the domain $G_A\{0 < x < A,\ 0 < \psi < \infty\}$, a solution of the equation*

$$\frac{\partial w}{\partial x} + v_0(x)\frac{\partial w}{\partial \psi} = \nu\sqrt{w}\,\frac{\partial^2 w}{\partial \psi^2} - 2\frac{dp(x)}{dx} \tag{4}$$

with conditions:

$$w|_{\psi=0} = 0, \qquad w|_{x=0} = w_0(\psi), \qquad w_0\left(\int_0^y u_0(y)\,dy\right) \equiv u_0^2(y), \tag{5}$$

which has the following properties: $w(x,\psi)$ is continuous and bounded in $\bar{G}_A$; $w > 0$ for $\psi > 0$; $w(x,\psi)$ has continuous derivatives entering

in equation (4); $|\partial w/\partial\psi| \le M$, $|\sqrt{w}\,\partial^2 w/\partial\psi^2| \le M$ in G_A; $|\partial w/\partial x| \le M\psi^{1-\beta}$, $\partial w/\partial\psi > m > 0$ for $0 \le \psi \le \psi_0$, $0 < \beta < 1/2$. The constants M, m, ψ_0 depend on A, v_0, u_0, p.

The assertion of the lemma follows from the fact that the change of variables

$$x' = x, \qquad \psi = \psi(x,y), \qquad w = u^2, \tag{6}$$

where $\partial\psi/\partial y = u$, $-\partial\psi/\partial x = v - v_0(x)$, $\psi(x,0) = 0$, reduces system (1) with conditions (2) to problem (4), (5) (see ([2])).

The construction of the solution of problem (4), (5) with the required properties is carried out as follows. Consider the domain $G_A^\varepsilon\{0 < x < A,\ 0 < \psi < 1/\varepsilon\}$. By Γ_A^ε denote its boundary lying on the straight lines $x = 0$, $\psi = 0$, $\psi = 1/\varepsilon$. In the domain G_A^ε consider equation (4) with the conditions on Γ_A^ε:

$$\begin{gathered} w|_{\psi=0} = w_0(\varepsilon)e^{V(\varepsilon)x/w_0(\varepsilon)}, \qquad w|_{x=0} = w_0(\varepsilon+\psi), \\ w|_{\psi=1/\varepsilon} = w_0(\varepsilon+1/\varepsilon)e^{V(\varepsilon+1/\varepsilon)x/w_0(\varepsilon+1/\varepsilon)}, \end{gathered} \tag{7}$$

where $V(\psi) = v\sqrt{w_0(\psi)}\,w_0''(\psi) - 2p'(0) - v_0 w_0'(\psi)$. According to condition (3), $V(\psi) = O(\psi)$ for small ψ, and $w \ge k\varepsilon$ on Γ_A^ε, where k does not depend on ε. If a positive solution $w_\varepsilon(x,\psi)$ of problem (4), (7) exists in the domain G_A^ε, then for $x \le x_0$ the a priori estimate

$$w_\varepsilon(x,\psi) \ge w_\varepsilon(x,0) + f(\psi)(1 + e^{-\alpha x}) \tag{8}$$

holds, where $\alpha > 0$, $f(\psi) = A_1\psi^{4/3} + A_2\psi$ for $\psi \le 1$, $A_1 > 0$, $A_2 > 0$; $A_3 > f(\psi) \ge f(1)$, $|f'(\psi)| \le A_4$, $|f''(\psi)| \le A_5$ for $\psi > 1$. The constants

$A_1, A_2, A_3, 1/\alpha$, and x_0 are sufficiently small and do not depend on ε. If $dp/dx \le 0$ and $v_0(x) \le 0$, then the estimate

$$w_\varepsilon(x,\psi) \ge w_\varepsilon(x,0) + f(\psi)e^{-\alpha x} \tag{9}$$

holds throughout the entire domain G^ε_A for any $A < \infty$ and $\varepsilon > 0$.

It is clear that, by virtue of the maximum principle, $w_\varepsilon \le A_6$ in G^ε_A, where A_6 also does not depend on ε. Using these estimates and theorem 13 of (^{3}), we obtain the existence of solutions of problem (4), (7) in the domain G^ε_A for $A = x_0$. If $dp/dx \le 0$, then w_ε exists in any domain G^ε_A for $A < \infty$ and $\varepsilon > 0$.

The solution of problem (4), (5) in G_A, $A = x_0$, is obtained as the limit as $\varepsilon \to 0$ of the solutions $w_\varepsilon(x,\psi)$ of problem (4), (7). For this it is sufficient to establish the following estimates for w_ε for $0 \le x \le A_0$:

$$0 < M_1 \le \partial w_\varepsilon(x,0)/\partial\psi, \qquad |\partial w_\varepsilon/\partial\psi| \le M_2, \qquad |\partial w_\varepsilon/\partial x| \le M_3,$$

$$|\sqrt{w_\varepsilon}\partial^2 w_\varepsilon/\partial\psi^2| \le M_4, \qquad |w_\varepsilon^{\beta-1}\partial w_\varepsilon/\partial x| \le M_5, \qquad 1/2 > \beta > 0,$$

where the constants M_i and β do not depend on ε, and to show that the norm of w_ε in $C^{2+\gamma}$ is bounded by a constant depending only on ψ_1, for $\psi \ge \psi_1 > 0$.

The estimates (8) and (9) are established with the aid of the maximum principle for equation (4). It follows from these estimates that $\partial w_\varepsilon(x,0)/\partial\psi \ge M_1 > 0$. Applying the maximum principle, we also obtain that

$$w_\varepsilon(x,\psi) - w_\varepsilon(x,0) \le (2\psi - \psi^{4/3})e^{\alpha(x+1)}$$

for $0 \le \psi \le 1$, and $\partial w_\varepsilon(x,0)/\partial\psi \le M_2'$, where $\alpha > 0$ does not depend on ε and is sufficiently large. Differentiating equation (4) with respect to ψ, we obtain an equation for $\partial w_\varepsilon/\partial\psi$, for which the maximum principle is valid. Therefore $|\partial w_\varepsilon/\partial\psi| \le M_2$ in G^ε_A. Taking this estimate, estimates (8) and (9) into account, and applying lemma 6 from (^{3}) to equation (4) for $\psi \ge \psi_1$, we obtain that w_ε satisfies the Hölder condition with constants independent of ε. Applying further the results of (^{4}), we obtain that $w_\varepsilon \in C^{2+\gamma}$ for $\psi \ge \psi_1$, with norm bounded uniformly in ε. The estimate $\partial w_\varepsilon/\partial x \ge -M_3$ is easily obtained by applying the maximum principle to the equation satisfied by $\partial w_\varepsilon/\partial x$.

It follows from equation (4) that

$$\sqrt{w_\varepsilon}\,\frac{\partial^2 w_\varepsilon}{\partial\psi^2} > -M_4.$$

A uniform in ε estimate for $\sqrt{w_\varepsilon}\,\partial^2 w_\varepsilon/\partial\psi^2$ from above holds on the straight lines $\psi = 0,\ x = 0,\ \psi = \psi_1$. If ψ_1 is sufficiently small, then

$$\frac{\partial w_\varepsilon(x,\psi)}{\partial\psi} \geqslant M_1 - M_4\int_0^\psi w_\varepsilon^{-1/2}\,d\psi \geqslant M_6 > 0$$

for $0 \leqslant \psi \leqslant \psi_1$. We differentiate equation (4) with respect to ψ and make the substitution $\partial w_\varepsilon/\partial\psi = \varphi(q_\varepsilon)$. We differentiate the resulting equation for q_ε with respect to ψ and put $\sqrt{w_\varepsilon}\,\partial q_\varepsilon/\partial\psi = \eta_\varepsilon$. If $\varphi(q_\varepsilon) = (1 + e^{q_\varepsilon})M_6/2$, then for $0 \leqslant \psi \leqslant \psi_1$ we obtain for η_ε an equation for which the maximum principle is valid. Therefore $\eta_\varepsilon \leqslant M_7$ and $\sqrt{w_\varepsilon}\,\partial^2 w_\varepsilon/\partial\psi^2 \leqslant M_4$. From (4) it follows that $|\partial w_\varepsilon/\partial x| \leqslant M_3$. To estimate $w_\varepsilon^{\beta-1}\partial w_\varepsilon/\partial x$, in equation (4) we make the substitution $w_\varepsilon^\beta = z_\varepsilon$ and differentiate it with respect to x. The equation for $\partial z_\varepsilon/\partial x$ has the maximum principle for $0 \leqslant \psi \leqslant \psi_2$, where ψ_2 does not depend on ε and is sufficiently small. Therefore

$$|\partial z_\varepsilon/\partial x| \leqslant M_5\beta.$$

It is easy to verify that $w(x,\psi) = \lim w_{\varepsilon_n}(x,\psi)$ as $\varepsilon_n \to 0$ is the required solution of (4), (5). Let us prove that $w \to U^2(x)$ as $\psi \to \infty$. Since $w_0(\psi) \to U^2(0)$ as $\psi \to \infty$, then $s(x,\psi) \equiv w(x,\psi) + 2p(x) - C \to 0$ for $x = 0$ and $\psi \to \infty$; $\nu\sqrt{w}\,\partial^2 s/\partial\psi^2 = \partial s/\partial x + v_0\partial s/\partial\psi$. On the basis of the maximum principle for the functions $\pm s + \delta + Me^{-\psi+\alpha x} \geqslant 0$ for $\psi \geqslant 1$, where $\delta > 0$ is arbitrary and $M > 0$ and $\alpha > 0$ are sufficiently large. Hence it follows that $|w(x,\psi) - U^2(x)| < 2\delta$ for large ψ.

Thus, the theorem on the existence of a solution of problem (1), (2) is proved.

2. Uniqueness theorem. Let u, v satisfy system (1) at points of D_A, be continuous in $\overline{D}_A$, and satisfy conditions (2); $0 < u < k_0$ for $\psi > 0$. Suppose the inequalities

$$k_1 y \geqslant u \geqslant k_2 y \quad \text{for } 0 \leqslant y \leqslant y_0, \quad k_1 > 0, \quad k_2 > 0; \quad |\partial^2 u/\partial y^2| \leqslant k_3 \text{ in } D_A. \tag{10}$$

are satisfied.

The solution u, v of problem (1), (2) possessing these properties is unique. In the case $dp/dx \leqslant 0$ and $v_0(x) \leqslant 0$, assumptions (10) may be omitted.

The proof of this theorem, by means of transformation (6), reduces to the proof of uniqueness of a solution of problem (4), (5) such that w is continuous and bounded in $\overline{G}_A$; $k_4\psi \geqslant w \geqslant k_5\psi$ for $0 \leqslant \psi \leqslant 1$; $|\sqrt{w}\,\partial^2 w/\partial\psi^2| \leqslant k_6$ in G_A, and $w > 0$ for $\psi > 0$. The difference of two such solutions w_1 and w_2 satisfies the equation

$$\nu\sqrt{w_1}\,\frac{\partial^2(w_1 - w_2)}{\partial\psi^2} - \frac{\partial(w_1 - w_2)}{\partial x} - v_0(x)\frac{\partial(w_1 - w_2)}{\partial\psi} + c(x,\psi)(w_1 - w_2) = 0,$$

where

$$c(x,\psi) = \left(\sqrt{w_1} + \sqrt{w_2}\right)^{-1} \frac{\partial^2 w_2}{\partial \psi^2}.$$

By virtue of the assumptions, $|c(x,\psi)(w_1 - w_2)| \leqslant k_7$, $|c(x,\psi)| \leqslant k_8\psi^{-1}$ for $0 < \psi \leqslant 1$, and $|c(x,\psi)| \leqslant k_9$ for $\psi \geqslant 1$.

Let $c_\delta \equiv c(x,\psi)$ for $\psi \geqslant \delta > 0$ and $c_\delta \equiv 0$ for $\psi \leqslant \delta$;

$$\mathcal{L}_\delta(F) \equiv \nu\sqrt{w_1}\frac{\partial^2 F}{\partial \psi^2} - \frac{\partial F}{\partial x} - v_0\frac{\partial F}{\partial \psi} + c_\delta F;$$

$\Phi = e^{\alpha x}\varphi(\psi)$, where $\alpha > 0$,

$$\varphi(\psi) = 2\psi - \psi^{4/3} \quad \text{for } 0 \leqslant \psi \leqslant 1,$$

$\varphi(1) \leqslant \varphi(\psi) \leqslant 2$ for $\psi > 1$, and φ', φ'' are bounded. It is easy to verify that $\mathcal{L}_\delta(\Phi) < -k_{10}\psi^{-1/6}$, $k_{10} > 0$, for $\psi \leqslant \psi_3$, and $\mathcal{L}_\delta(\Phi) < 0$ for $\psi \geqslant \psi_3$. Let $\varepsilon > 0$ be arbitrary. For sufficiently small δ,

$$\mathcal{L}_\delta(\varepsilon\Phi \pm (w_1 - w_2)) < 0.$$

Therefore $|w_1 - w_2| \leqslant \varepsilon\Phi$, and consequently $w_1 \equiv w_2$.

In the case $dp/dx \leqslant 0$ and $v_0(x) \equiv 0$, the existence of merely a solution of problem (4), (5) continuous in $\overline{D}_A$ was obtained in paper (5). Questions of uniqueness of solutions of problems for the boundary-layer equations were studied in the works of Nickel (see (6)).

3. Behavior of the solutions of problem (4), (5) as $x \to \infty$ in the case $dp/dx \leqslant 0$. Let $2P(x) = -2(p(x) - p(0)) + K_0$. The following estimate holds:

$$\int_0^\infty \left|\sqrt{w_1(X,\psi)} - \sqrt{w_2(x,\psi)}\right| d\psi \leqslant \frac{\sqrt{P(0)}}{\sqrt{P(X)}} \int_0^\infty \left|\sqrt{w_1(0,\psi)} - \sqrt{w_2(0,\psi)}\right| d\psi, \tag{11}$$

where $K_0 = \max_\psi\{w_1(0,\psi), w_2(0,\psi)\}$, for any two solutions $w_1(x,\psi)$ and $w_2(x,\psi)$ of problems of the form (4), (5). In the case $v_0(x) \equiv 0$, using (11) and the particular solutions (1), (2) constructed in $(^2, ^7)$, one can show that

$$\left|\sqrt{w_1(X,\psi)} - \sqrt{w_2(X,\psi)}\right| \leqslant K_1(X+1)^{-1/2},$$

if $|dp/dx| \geqslant K_2(1+x)^{-1/3}$ and the integral on the right-hand side of (11) converges, $K_i = \text{const}$. If

$$|dp/dx| \leqslant K_3(1+x)^{-(1/3+\varepsilon)}, \qquad \varepsilon > 0,$$

then $w(x, \psi) \to 0$ as $x \to \infty$, uniformly on every finite interval of the ψ-axis.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.