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Abstract

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ON THE THEORY OF COMPENSATING FIELDS

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1. As is known, ordinary tensor analysis is based on the local group of linear transformations. This paper considers the behavior of tensor quantities and connection coefficients in an orthonormal frame with respect to the local Lorentz group or, in other words, with respect to a 6-parameter group with parameters depending on the coordinates of space-time; for this purpose the method of compensating fields is used (^{1,2}). In the present work, as in (^{1}), the nontensor character of fields and their transformation properties under finite transformations of the local group are considered, which gives the arguments greater rigor, since Lie' s theorem does not apply to local groups.

2. The Ricci coefficients $\Delta_\sigma(l,k) = \Omega^\tau(l)\Omega^\alpha(k)\Delta_{\sigma,\tau\alpha}$; $\Delta_\sigma(l,m) = -\Delta_\sigma(m,l)$, $\Gamma_{\lambda,\sigma\tau} = \Omega_\lambda(\beta)\partial_\sigma\Omega_\tau(\beta) + \Delta_{\sigma,\tau\lambda}$, where $\partial_\sigma = \partial/\partial u_\sigma$; u are world coordinates; $\Omega_\alpha(l)$ are the Lamé coefficients, transform under rotations of the orthonormal frame: $\Omega'_\sigma(i)\exp[\varepsilon_{lk} M^{kl}_{sj}]\Omega_\sigma(s)$, where $\varepsilon_{lk} = \varepsilon_{lk}(u)$,

$$M^{kl}_{sj} = {}^1/_2 \left|(\delta_{ls}\delta_{kj} - \delta_{ks}\delta_{lj})\right|. \tag{1}$$

Then:

$$\begin{aligned}\Delta'_{\sigma,\tau\lambda} = \Delta_{\sigma,\tau\lambda} + {}^1/_2 M^{sp}_{mn} N^{nm}_{ji} \{&\Omega_\sigma(s)(\Omega_\lambda(p)\partial_\tau - \Omega_\tau(p)\partial_\lambda) \\ &+ \Omega_\lambda(s)(\Omega_\sigma(p)\partial_\tau - \Omega_\tau(p)\partial_\sigma) + \Omega_\tau(s)(\Omega_\lambda(p)\partial_\sigma - \Omega_\sigma(p)\partial_\lambda)\}\varepsilon_{ij}(u).\end{aligned} \tag{2}$$

Hence, taking into account that

$$M^{sf}_{mn} = -M^{fs}_{mn},$$

$$\Delta'_\sigma(f,g) = \mathcal{L}^{-1}_{ff'}\mathcal{L}^{-1}_{gg'}\Delta_\sigma(f',g') + {}^1/_2 \mathcal{L}^{-1}_{ff'}\mathcal{L}^{-1}_{gg'} M^{sp}_{mn} N^{nm}_{ji} (\delta_{f'p}\delta_{g's} - \delta_{f's}\delta_{g'p})\partial_\sigma\varepsilon_{ij};$$

$$N^{nm}_{ji} = \int_0^1 \exp[\, t\varepsilon_{lk} c^{nm}_{kl,ji}\,]\, dt,$$

the world coordinates u_σ, related to the frame coordinates by the formula $x_i = \Omega_\sigma(i)u_\sigma$, obviously do not change under simultaneous rotations: $x_i = \mathcal{L}_{ii'}x_{i'}$, $\Omega'_\sigma(i) = \mathcal{L}_{ii'}\Omega_\sigma(i')$, where $\mathcal{L}_{ii'} = \exp[\varepsilon_{lk}M^{kl}_{ii'}]$. The expression N^{nm}_{ji} is obtained, as shown in ([1]), by expanding $S\partial_\sigma S^{-1}$ according to the formula from ([3]), where S is a representation of the local Lorentz group (in the particular case it coincides with M^{kl}_{sj}), of the form $S = \exp[\varepsilon_{ij}I_{ji}]$, with $[I_{ik}, I_{jl}] = c^{pm}_{ik,jl}I_{pm}$.

The structure constants of the group, determining the commutation relations for the infinitesimal operators of the group, have the form:

$$c^{ml}_{ij,sp} = \delta_{js}\delta_{im}\delta_{pl} + \delta_{jp}\delta_{sm}\delta_{il} + \delta_{is}\delta_{pm}\delta_{jl} + \delta_{ip}\delta_{sm}\delta_{il}.$$

3. We now show that the general form of the gravitational interaction can be obtained without using the concept of parallelism, if one requires covariance of the equation for a wave field Ψ ([4]), written

for convenience, in world coordinates.

$$(\Omega_\sigma(p)L_p\partial_\sigma + im)\Psi = 0 \tag{3}$$

with respect to $\mathcal{L}_{ii'}$, the wave function transforming according to the group representation: $\Psi' = S\Psi$. Then, if the matrices L_p satisfy the covariance conditions $\mathcal{L}_{pp'}SL_{p'}S^{-1} = L_p$, we see that, in order to compensate the term $S\partial_\sigma S^{-1} = I_{ml}N^{lm}_{ps}\partial_\sigma\varepsilon_{sp}(u)$, whose appearance is due to the locality of the group, it is necessary to replace ∂_σ by the covariant derivative $\nabla_\sigma = \partial_\sigma - \Gamma_\sigma$, where

$$\Gamma_\sigma = \frac{1}{2}I_{mn}\Delta_\sigma(m,n). \tag{4}$$

The gravitational interaction specified by the compensating field Γ_σ transforms under rotations of the frame according to the formula

$$S\Gamma'_\sigma S^{-1} = \Gamma_\sigma + I_{mn}N^{nm}_{ji}\partial_\sigma\varepsilon_{ij}, \tag{5}$$

taking into account that $\mathcal{L}_{pp'}\mathcal{L}_{kk'}SI_{p'k'}S^{-1} = I_{pk}$ (the last equality follows from the expression given above for the structure constants of the Lorentz group). In the case of spinor fields described by the Dirac equation, equation (3) has the form

$$(\Omega_\sigma(p)\gamma_\sigma(\partial_\sigma - \Gamma_\sigma) + im)\Psi = 0, \tag{6}$$

where $\Gamma_\sigma = \frac{1}{4}[\gamma_l\gamma_m]\Delta_\sigma(m,l)$. The expression Γ_σ was obtained in (^{5}) for parallel transport of spinors and, as we see, follows directly from (4) for $L_i = \gamma_i$, since in this case $I_{ik} = \frac{1}{2}[\gamma_i\gamma_k]$.

4. As already noted, the local Lorentz group is an infinite-parameter group, and consequently, generally speaking, Lie' s theorem does not apply to it, according to which a continuous group is specified by infinitesimal transformations I_{lk} in a neighborhood of the identity. In other words, to justify the use in this case of the formula $S = \exp[\varepsilon_{lk}, I_{kl}]$, we shall show that Γ_σ retains its form also for infinitesimal rotations of the frame $\Omega'_\nu(l) = \mathcal{L}_{ll'}\Omega_\nu(l')$, where $\mathcal{L}_{ll'} = \exp[\varepsilon_{pk}M^{kp}_{ll'}]$ goes over into $\mathcal{L}_{ll'} = \delta_{ll'} + \varepsilon_{ll'}$. Then

$$\Delta'_{\sigma,\tau\alpha} = \Delta_{\sigma,\tau\alpha} + \frac{1}{2}[\Omega_\alpha(j)(\Omega_\sigma(s)\partial_\tau - \Omega_\tau(s)\partial_\sigma) + \\ + \Omega_\sigma(j)(\Omega_\alpha(s)\partial_\tau - \Omega_\tau(s)\partial_\alpha) + \Omega_\tau(j)(\Omega_\alpha(s)\partial_\sigma - \Omega_\sigma(s)\partial_\alpha)]\varepsilon_{ij}(u),$$

$$\Delta'_\sigma(l,k) = \mathcal{L}_{ll'}\mathcal{L}_{kk'}\Delta_\sigma(l',k') - M^{kl}_{sj}\partial_\sigma\varepsilon_{js}(u),$$

if one takes into account that the formula $\Gamma_{\lambda,\sigma\tau} = \Omega_\lambda(\beta)\partial_\sigma\Omega_\tau(\beta) + \Delta_{\sigma,i\tau}\lambda$ is equivalent to the expression

$$\Delta_{\sigma,\tau\alpha} = \frac{1}{2}[\Omega_\alpha(j)(\partial_\tau\Omega_\sigma(j) - \partial_\sigma\Omega_\tau(j)) + \Omega_\sigma(j)(\partial_\tau\Omega_\alpha(j) - \partial_\alpha\Omega_\tau(j)) + + \Omega_\tau(j)(\partial_\sigma\Omega_\alpha(j) - \partial_\alpha\Omega_\sigma(j))]$$

(see (^{7})). Hence

$$S\Gamma'_\sigma S^{-1} = \Gamma_\sigma - \frac{1}{2}I_{kl}\partial_\sigma\varepsilon_{kl}(u);$$

$$\Omega_\alpha(i)\Omega_\beta(i) = g_{\alpha\beta};$$

$$\sum_i \Omega^\sigma(i)\Omega_\tau(i) = \delta^\sigma_\tau.$$

We see that the equation $(\Omega_\sigma(p)L_p\nabla_\sigma + im)\Psi = 0$ preserves its covariance, since

$$(L_{p'}\Omega_\sigma(p')\partial_\sigma + im + L_{p'}\Omega_\sigma(p')\Gamma_\sigma +$$

$$+\frac{1}{2}L_{p'}\Omega_\sigma(p')I_{sj}\partial_\sigma\varepsilon_{js} - \frac{1}{2}L_{p'}\Omega_\sigma(p')I_{sj}\partial_\sigma\varepsilon_{js})\Psi = 0,$$

$$SI_{kl}S^{-1}\partial_\sigma\varepsilon_{kl} = I_{kl}\partial_\sigma\varepsilon_{kl}.$$

The compensating field, obviously, is determined only up to an arbitrary tensor quantity, since the compensation occurs precisely at the expense of the nontensorial behavior of the field under local transformations $\Omega_{ll'}$. Thus, the method of compensating fields does not make it possible to find explicitly the tensor part of the connection coefficients Γ_σ. To find the latter it is necessary to introduce the concept of parallelism. In this sense the torsion of space, defined by the tensor coefficients of the connection, cannot be introduced by the method of compensating fields.

In this connection it is interesting to note the considerations of [6], according to which the condition of local gauge invariance of the equations $(\Gamma_i\partial_i + im)\Psi = 0$ with respect to $\Psi' = \exp[i\alpha]\Psi$ does not, strictly speaking, introduce an electromagnetic field, since only the longitudinal (unphysical) part of the field contributes to the gradient transformations $A'_i = A_i + \partial_i\alpha(x)$, introducing the compensating term $\partial_i\alpha$, which is a consequence of the locality of the gauge group. In this case the transverse part of the field behaves as a tensor and is not determined from the transformation $\Psi' = \exp[i\alpha]\Psi$.

Returning to the question of the compensating treatment of the gravitational interaction, one may say that from local invariance one can derive only the connection coefficient Γ_i (4), analogous to the longitudinal component of the electromagnetic field. Hence the incorrectness is evident of the assertion in [6], according to which the method of compensating fields makes it possible to determine only the unphysical parts of interaction fields, since the nontensorial field is precisely the field that has physical meaning.

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