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Abstract

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MATHEMATICS

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ON THE SOLVABILITY OF THE FIRST BOUNDARY-VALUE PROBLEM FOR QUASILINEAR EQUATIONS WITH RAPIDLY GROWING COEFFICIENTS IN ORLICZ CLASSES

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In ([1],[2]) we considered the first boundary-value problem for quasilinear equations and systems of order $2m$ with "coefficients" having only a power order of growth with respect to the derivatives $D^\alpha u$, $|\alpha| = m$. Here we shall consider the case when the "coefficients" may have an arbitrary order of growth with respect to $D^\alpha u$. If this order of growth is greater than any power, then the solution of the first boundary-value problem is found in a certain Orlicz class ([3]), related to the principal part of the equation in the same way as the Dirichlet integral is related to the principal part of linear elliptic equations.

Let in a bounded domain $G \subset R^n$ with boundary Γ there be given an elliptic equation (respectively, a system of equations) of order $2m$:

$$L(u) \equiv L_0(u) + V(u) = \sum_{|\alpha|=m} (-1)^m D^\alpha A_\alpha(x, D^\omega u, D^\gamma u) + V(u) = 0, \quad (1)$$

where $|\omega| < m;\ |\gamma| = m;\ D^\beta = \partial^{|\beta|}/\partial^{\beta_1}x_1 \ldots \partial^{\beta_n}x_n;\ |\beta| = \beta_1 + \ldots + \beta_n;$
$L_0(u)$ is the principal part of the operator $L(u)$;

$$V(u) \equiv \sum_{|\beta|<m} (-1)^{|\beta|} D^\beta V_\beta(x, D^\omega u, D^\gamma u) + \sum_{|\beta|=m} (-1)^m D^\beta V_\beta(x, D^\omega u) \quad (2)$$

is an operator of order not exceeding $2m-1$, which, under the conditions formulated below, we shall call subordinate. On Γ, for simplicity, homogeneous boundary conditions are prescribed:

$$u|_\Gamma = 0, \qquad D^\omega u|_\Gamma = 0, \qquad |\omega| \le m-1. \quad (3)$$

(For nonhomogeneous boundary conditions see ([2]).)

We formulate four conditions under which the solvability of problem (1), (3) is proved.

Condition 1. For any K and $|\xi_\omega| \le K,\ |\omega| \le m-1$, there exist constants $C,\ c,\ C_1$ and a convex N-function $\rho(\zeta)$, $\zeta = \left(\sum_{|\alpha|=m} \zeta_\alpha^2\right)^{1/2}$, depending on K, such that for large ζ

$$c\rho(\zeta) \le \sum_{|\alpha|=m} A_\alpha(x, \xi_\omega, \zeta_\gamma)\zeta_\alpha \le C\rho(\zeta). \tag{4}$$

In addition, it is assumed that:

a) for any $p > 0$

$$\rho(\zeta) > c_p|\zeta|^p - C_p$$

(the case of power growth of $\rho(\zeta)$ was considered in ([1],[2])); $\rho'(0) = 0$, $\dfrac{d}{d\zeta}\left(\dfrac{\rho(\zeta)}{\zeta}\right) > 0$ for $\zeta > 0$;

b)

$$|A_\alpha(x, \xi_\omega, \zeta_\gamma)| < C\,\frac{\rho(\zeta)}{\zeta} + C_1 \quad (|\xi_\omega| \le K,\ x \in G); \tag{5}$$

c) the functions $A_\alpha(x, \xi_\omega, \zeta_\gamma)$ are continuous in all arguments and continuously differentiable with respect to ζ_γ; with respect to the arguments x, ξ_ω they are continuous.

in the following sense:

$$\left|A_\alpha(x, \xi_\omega, \zeta_\gamma) - A_\alpha(x', \xi'_\omega, \zeta_\gamma)\right| \le \varepsilon\left(|x - x'| + |\xi - \xi'|\right)\frac{\rho(\zeta)}{\zeta}, \tag{6}$$

where $\varepsilon(t) \to 0$ as $t \to 0$.

Condition 2. For $|\xi_\omega| \le K,\ |\omega| \le m-1$, there exists a constant $C = C(K)$ such that

$$\sum_{|\alpha|,|\beta|=m} A_{\alpha\beta}(x, \xi_\omega, \zeta_\gamma)\eta_\beta\eta_\alpha \ge C\left(\frac{\rho(\zeta)}{\zeta^2+1} + 1\right)\sum_{|\alpha|=m}\eta_\alpha^2, \tag{7}$$

where $A_{\alpha\beta} = \partial A_\alpha/\partial\zeta_\beta$. (This condition means that the variation of the operator L_0 with respect to $D^\gamma u$ is a positive definite operator under the boundary conditions (3).)

Condition 3 (it ensures the subordination of the operator $V(u)$). For $|\xi_\omega| \leq |K|$, $|\omega| \leq m-1$, there exist functions $f_\beta(x)$ and $q_\beta(x)$, depending on K, such that

$$\text{for } |\beta| = m \qquad |V_\beta(x, \xi_\omega)| \leq f_\beta(x), \tag{8}$$

where $[N(f_\beta(x))] < +\infty$, $N(\eta)$ is the complementary convex N-function to $\rho(\zeta)$ (recall that $\xi\eta \leq \rho(\xi) + N(\eta)$; $[\varphi]$ is the integral of φ over G);

$$\text{for } |\beta| < m \qquad |V_\beta(x, \xi_\omega, \zeta_\gamma)| \leq q_\beta(x)\rho_1(\zeta) + q_\beta(x) f_\beta(x), \tag{9}$$

where $\rho_1(\zeta) > 0$ for $\zeta > 0$ and $\lim \rho_1(\zeta)/\rho(\zeta) = 0$ as $\zeta \to \infty$; $q_\beta(x) \geq 0$ and may have singularities on Γ at a finite number of points or manifolds. For example, in the case of a singularity at one point $x_0 \in \Gamma$,

$$q_\beta(x) = \frac{C}{|x - x_0|^k}, \qquad \text{where } k < m - |\beta|. \tag{10}$$

Condition 4. For any ξ_ω, ζ_γ the estimate holds

$$\sum_{|\beta|<m} V_\beta(x, \xi_\omega, \zeta_\gamma)\xi_\beta + \sum_{|\beta|=m} V_\beta(x, \xi_\omega)\zeta_\gamma \leq$$

$$\leq (1-\delta) \sum_{|\alpha|=m} A_\alpha(x, \xi_\omega, \zeta_\gamma)\zeta_\alpha + f(x), \tag{11}$$

where $f(x) \in \mathcal{L}_1$, $\delta > 0$; moreover, it is assumed that

$$\sum_{|\alpha|=m} A_\alpha(x, \xi_\omega, \zeta_\gamma)\zeta_\alpha \geq C_2 \sum_{|\gamma|=m} |\zeta_\gamma|^{n+\varepsilon} - C_1, \qquad \varepsilon > 0. \tag{12}$$

We note that from (11) and (12) there follows the uniform boundedness of possible solutions of (1), (3):

$$\sum_{|\gamma|=m} [|D^\gamma u|^{n+\varepsilon}] \leq C_3$$

and, consequently,

$$|D^\omega u| \leq K \qquad \text{for } |\omega| \leq m-1. \tag{13}$$

To verify this, it suffices to multiply (1) scalarly by u, and to use first (11) and then (12).

Here is the simplest example of an equation for which all the conditions listed above are satisfied:

$$-\sum \frac{\partial}{\partial x_i}\left(\exp\left[\sum\left(\frac{\partial u}{\partial x_k}\right)^2\right](1+\varphi_i(x,u))\frac{\partial u}{\partial x_i}\right)+$$

$$+\sum \frac{\psi_i(x,u)}{|x-x_0|^{1-\varepsilon}}\exp\left[\sum\left(\frac{\partial u}{\partial x_k}\right)^2\right]\sum\left|\frac{\partial u}{\partial x_k}\right|^{2-\varepsilon_1}+\sum\frac{\partial h_i(x)}{\partial x_i}=0,$$

where $\varepsilon, \varepsilon_1 > 0,\ h_i(x) \in \mathcal{L}_{1+\varepsilon_2},\ \varepsilon_2 > 0,\ \varphi_i(x,u),\ \psi_i(x,u) \geq 0,\ i = 1, \dots, n.$

By the Orlicz class $O_\rho^{(m)}$ we shall understand the set of functions $u(x)$ satisfying the conditions (3), for which the integral

$$[\rho(D^m u)] < +\infty, \qquad D^m u = \left(\sum_{|\alpha|=m} |D^\alpha u|\right)^{1/2}. \tag{14}$$

Let $\rho(\zeta)$ correspond to the constant K in (13) and to Conditions 1, 2. A function $u(x)$ is called a **solution** of problem (1), (3) in the class $O_\rho^{(m)}$ if $u \in O_\rho^{(m)}$, $|D^\omega u| \leq K$ for $|\omega| \leq m-1$, and for every function $v \in O_\rho^{(m)}$ (and their linear hull, forming the Orlicz space $H_\rho^{(m)}$) ful-

the relation

$$\sum_{|\alpha|=m}[A_\alpha(x, D^\omega u, D^\gamma u), D^\alpha v] + \sum_{|\beta|\leqslant m}[V_\beta(\dots), D^\beta v] = 0. \tag{15}$$

From conditions 1-4 there follows the convergence of all integrals in (15).

Theorem. *If the functions A_α and V_β in equation (1) satisfy conditions 1-4, then problem (1), (3) is solvable in the class $O_\rho^{(m)}$.*

We give the idea of the proof.

1. First one proves the solvability of an equation of the form

$$L(u) \equiv \sum_{|\alpha|=m}(-1)^m D^\alpha A_\alpha(x, D^\gamma u) + h(x) = 0 \tag{16}$$

under the boundary conditions (3); here the A_α satisfy conditions 1, 2 and the A_α are continuously differentiable with respect to x, and moreover

$$|\partial A_\alpha(x, \xi_\gamma)/\partial x_i| \leqslant C\frac{\rho(\xi)}{\xi} + C_1, \qquad h(x) \in \mathcal{L}_2(G).$$

The proof of the existence of a solution of this problem is carried out with the aid of an analogue of the Galerkin method described in ([1,2]). Let $\{z_i(x)\}$ be a system of smooth functions such that the functions

$$v_i(x) = Bz_i \equiv -\psi(x)\Delta z_i + Mz_i$$

$$(M > 0,\ \psi(x) > 0,\ x \in G,\ \psi|_\Gamma = D^\nu \psi|_\Gamma = 0,\ |\nu| \leqslant 2m-1)$$

form a complete system in $C^{(m)}(\overline{G})$. The approximate solution $u_k = \sum C_{ki} z_i$ $(i = 1, \dots, k)$ of problem (1), (3) is determined from the system of nonlinear equations:

$$[L(u_k), Bz_j] = 0 \qquad (j = 1, \dots, k). \tag{17}$$

From the analogue of Lemma 2 ([2]) there follows the solvability of system (17) with respect to C_{ki}. From the relation $[L(u_k), Bu_k] = 0$, which follows from (17), with the aid of conditions 1, 2 we derive the estimate

$$[\rho(D^m u_k)] + [\psi(x)\left(\rho(D^m u_k)/(|D^m u_k|^2 + 1) + 1\right)|D^{m+1} u_k|^2] < C, \tag{18}$$

where C does not depend on k. Hence it follows that there exists a subsequence $\{u_r\}$ such that $\{u_r\}$ and $\{D^\gamma u_r\}$, $|\gamma| = m$, converge almost everywhere in G to u and, respectively, to $D^\gamma u$. From (18), (5), and the Vallée-Poussin theorem it follows that $A_\alpha(x, D^\gamma u_r)$ are equi-absolutely continuous (with respect to r) functions of x. Substituting $k = r$ in (17) and passing to the limit as $r \to \infty$, we obtain, using the completeness of $\{Bz_j\}$, that

$$\sum_{|\alpha|=m} [A_\alpha(x, D^\gamma u), D^\alpha v] + [h, v] = 0 \tag{19}$$

for any $v \in C^{(m)}(\overline{G})$. Next, by means of a process of weak closure with respect to v, the validity of relation (19) is established for any $v \in O_\rho^{(m)}$ (and $v \in H_\rho^{(m)}$).

The function $u(x)$ found is the unique solution of problem (16), (3), since, by virtue of conditions 2 and 1,

$$C_5\left[\rho\left(\frac{D^m(u_1 - u_2)}{C_4}\right)\right] \leqslant \sum_{|\alpha|=m} [A_\alpha(x, D^\gamma u_1) - A_\alpha(x, D^\gamma u_2), D^\alpha(u_1 - u_2)]. \tag{20}$$

2. It is established that problem (16), (3) is uniquely solvable if $A_\alpha(x, \xi_\gamma)$ are only continuous with respect to x and $h(x) = \sum D^\alpha h_\alpha(x)$, $|\alpha| \leqslant m$, where $[N(h_\alpha)] < +\infty$ (h is a generalized function); $N(\eta)$ is an additional function to $\rho(\xi)$. For the proof, equations of the form (16) are considered in which the functions A_α and $h_\alpha(x)$ are replaced by their averages with respect to x (see ([2])), and it is proved that the solutions of these averaged equations converge to the desired solution.

3. We now consider the general equation

$$L(u) = L_0(u) + V^\lambda(u) = 0, \tag{21}$$

where $L_0(u)$ is given by formula (1), and $V^\lambda(u)$ by formula (2), in which the func-

the functions V_β have been replaced by bounded functions V_β^λ:

$$V_\beta^\lambda = (1 + \lambda \Sigma V_{\beta_1}^2)^{-1} V_\beta \quad (|\beta_1| \leq m), \tag{22}$$

which tend, as $\lambda \to 0$*, to* V_β. Let us prove that problem (21), (3) has at least one solution $u_\lambda(x)$. To this end, in the principal part $L_0(u)$ (see (1)) we replace the arguments $D^\omega u$, $|\omega| \leq m-1$, by $D^\omega w$, where $w \in C^{(m-1)}(\overline{G})$:

$$L_0(w; u) \equiv \sum_{|\alpha|=m} (-1)^{m D^\alpha} A_\alpha(x, D^\omega w, D^\gamma u). \tag{23}$$

For the operator $L_0(w; u)$, for any fixed function $w \in C^{(m-1)}(\overline{G})$, the functions $A_\alpha(x, D^\omega w, \xi_\gamma)$ are continuous in x, and, by item 2, it is uniquely invertible: $L_0(w; u) = h$, $u = R_0(w; h)$. Equation (21) is equivalent to the following:

$$u = R_0(u, -V^\lambda(u)). \tag{24}$$

The operator $R_0(u, -tV^\lambda(u))$, considered, for example, in the space $u \in W_p^{(m)}$, where $p > n$, is completely continuous for any t, $0 \leq t \leq 1$, and moreover $R_0(u, -0 \cdot V^\lambda) \equiv 0$.

From condition 4 we infer that, for sufficiently large r, the degree of the covering of zero under the mapping of the ball $\|u\|_{m,p} \leq r$ by the operator $u \to u - R_0(u, -tV^\lambda(u))$ does not depend on t, $0 \leq t \leq 1$. Since for $t = 0$ it is equal to one, it is also equal to one for $t = 1$. Hence it follows that equation (24), and with it problem (21), (3), has at least one solution $u = u_\lambda(x)$, which, by conditions 4 and 1, belongs to $O_\rho^{(m)}$, and $|D^\omega u_\lambda| \leq K$.

4. Let us prove that some subsequence of the solutions found $u_\lambda(x)$ converges to a solution of the general equation (1), whose coefficients satisfy conditions 1–4. From conditions 1 and 4 it follows that, for any $\lambda > 0$,

$$|D^\omega u_\lambda| \leq K, \quad \text{for } |\omega| \leq m-1, \qquad [\rho(D^m u_\lambda)] \leq K_1. \tag{25}$$

Hence, from (21) for $u = u_\lambda$, from condition 3, and from (20), it follows that there exists a subsequence $\mu \to 0$ for which $D^\gamma u_\mu \to D^\gamma u$, $|\gamma| \le m$, $u_\mu \to u$ almost everywhere in G, and the functions $A_\alpha(x, D^\omega u_\mu, D^\gamma u_\mu)$, $V_\beta^\mu(x, D^\omega u_\mu, D^\gamma u_\mu)$ (for $|\beta| < m$), $V_\beta^\mu(x, D^\omega u_\mu)$ (for $|\beta| = m$) are equi-absolutely continuous (in μ) with respect to x.

It follows that in the relation satisfied by the functions u_μ:

$$\sum_{|\alpha|=m} [A_\alpha(x, D^\omega u_\mu, D^\gamma u_\mu), D^\alpha v] + \sum_\beta [V_\beta^\mu(...), D^\beta v] = 0, \qquad (26)$$

one may pass to the limit as $\mu \to 0$, if $v \in C^{(m)}(\overline{G})$. We obtain that the limiting function $u(x)$ satisfies relation (15) for $v \in C^{(m)}(\overline{G})$. By means of a limiting passage in v we establish the validity of (15) for arbitrary $v \in O_\rho^{(m)}$. The theorem is proved.

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CITED LITERATURE

1. M. I. Vishik, DAN, **138**, No. 3, 518 (1961).
2. M. I. Vishik, Tr. Moscow Math. Soc., **12**, 125 (1963).
3. M. A. Krasnosel' skii, Ya. D. Rutitskii, *Convex Functions and Orlicz Spaces*, Moscow, 1958.
4. T. B. Solomyak, DAN, **146**, No. 6, 1282 (1962).

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* We note that T. B. Solomyak (see, for example, (4)) used an analogous idea of replacing the coefficients A_α of the principal part by bounded A_α^λ in the case of equations of second order. Here this idea is used for replacing the lower-order coefficients V_β by bounded V_β^λ.

Note: Figure translations are in progress. See original paper for figures.