MATHEMATICS

1963

SovietRxiv

Abstract

Full Text

MATHEMATICS

Yu. G. BORISOVICH

ON ONE APPLICATION OF THE CONCEPT OF ROTATION OF A VECTOR FIELD

(Presented by Academician P. S. Aleksandrov, 5 VI 1963)

In note (^{1}) we studied the rotation of a weakly continuous vector field in hyperboloids (see also ($^{2-3}$)). The method used can be generalized and applied to weakly or completely continuous operators acting on certain invariant subsets of a Banach space. In the present note we set forth the results obtained in this direction; certain considerations were stated in (^{4}); one particular case was considered in (^{5}).

1. Let us consider a linear locally convex topological space E (l.c.s.), and let E_1 be a convex closed subset in it, which we regard as a topological space with the topology $\{U\}$ induced by the topology $\{U_E\}$ of the space E. Suppose that on the closure $\overline{U}$ an operator Fx is defined, acting in E_1 and completely continuous (F is continuous and $\overline{FU}$ is compact). Assume that the vector field $(I-F)x$, where I is the identity operator, does not vanish on the boundary $\dot{U}$. The following is true (see also (^{6})).

Lemma 1. *The sets $(I-F)\dot{U}$, $(I-F)\overline{U}$ are closed in E; there exists such a convex neighborhood $U^*_E(\theta)$ that $(I-F)\dot{U} \cap U^*_E(\theta)$ is empty.*

Denote the pseudonorm generated by $U^*_E(\theta)$ by $\|x\|$ and construct the Schauder projection operator

$$P_m x = \left(\sum r_i(x)x_i\right)\left(\sum r_i(x)\right)^{-1}, \qquad x \in E_1,$$

where $\{x_i\}_1^m$ is an ε-pseudonet ($\varepsilon < 1/3$) of the set $\overline{FU}$, and $r_i(x) = \varepsilon - \|x - x_i\|$, if $\|x - x_i\| < \varepsilon$, and $r_i(x) = 0$ otherwise ($i = 1, 2, \dots, m$).

Lemma 2. *The operators Fx and P_{mFx} are continuously homotopic in E_1 without fixed points on the boundary $\dot{U}$, and moreover $\|P_{mFx} - Fx\| < \varepsilon$, $x \in \overline{U}$.*

Since P_{mF} is a finite-dimensional operator, the rotation $\gamma(\dot{U})$ of the field $x - Fx$ on $\dot{U}$ is defined in the usual way (see (^{7,8})). Moreover, $\gamma(\dot{U})$ does not depend on the choice of the extension of the operator F to U, depends additively on the set U, is a homotopy invariant in the class of completely continuous vector fields on $\overline{U}$, and is equal to the sum of the indices of the fixed points in U, if their number is finite. We note that, in contrast to the works ($^{6,8-10}$), we consider an operator on an invariant set E_1.

Theorem 1. *If there exists a continuous deformation $\Phi_t x$ of the set $\overline{U}$ into one of its points $x \in U$, and $\Phi_t \overline{U} \subset U$, $t > 0$, $F\overline{U} \subset \overline{U}$, then there exists a fixed point of the operator F; if the rotation $\gamma(\dot{U})$ is defined, then it is equal to 1.*

Theorem 2. *Let a completely continuous operator F be defined on $\overline{U}$; let D be a convex closed subset in U, and suppose $F^{jD} \subset U$, $j = 1, 2, \dots, m-1$, $F^{mR} \subset D$ (F^j denotes the j-th iteration of F, $R -$*

part of the definition of F^m, $R \subset \overline{U}$); suppose that in E the convex closure of every compact set is compact.

Then $\gamma(\dot{U})$ (the rotation of the field $x - Fx$) is equal to 1; consequently, in D there is a fixed point of the operator F.

Let us note that Theorem 2 strengthens Brouwer' s theorem for Banach spaces (^{11}); it is also valid for convexoid spaces of Leray (^{12}).

Theorem 3. Let A be a convexoid space, U an open set, F a continuous mapping of $\overline{U}$ into A; D a closed set contractible to a point,

$$\bigcup_{0 \le j \le m-1} F^{jD} \subset U, \qquad F^{mR} \subset D.$$

Then the full index of the solutions $\gamma(\dot{U})$ is equal to 1; consequently, F has a fixed point.

We note that Theorem 2 follows from Theorem 3 if, as A, one considers the convex closure of the set FU; then the rotation $\gamma(\dot{U})$ coincides with the full index (^{12}) $\gamma(\dot{U})$ in A, where $\dot{U} = U \cap A$.

We give one variant of the theorem on an odd field (^{7,16,1}):

Theorem 4. Let E_1 be centrally symmetric with respect to θ, and let $\overline{U}$ be a set centrally symmetric and star-shaped with respect to θ, on which a completely continuous field is defined, odd on the boundary $\dot{U}$. Then the rotation of the field is odd.

2. In this section cones and wedges in the direct sum $E = B_1 \dot{+} B_2$ of Banach spaces are considered.

Let K_1 and K_2 be cones (^{13}) in B_1 and B_2. The direct sum $K = K_1 \dot{+} K_2$ will be a cone in E, and $W = K_1 \dot{+} B$ is a wedge (^{14}). By choosing in B_1, B_2 the weak or strong topology, we define in E a Tikhonov topology which will induce a topology in W or K.

Suppose first that in W (K) the topology of the norm $\|x\| = \|x_1\| + \|x_2\|$ is induced, and the operator F is completely continuous on $\overline{U}$ (for example, on the ball $\|x\| \le \rho$, $x \in W$ (K)) and acts in W (K). Then on the boundary $\dot{U}$ (on the sphere $\|x\| = \rho$, $x \in W\ (K)$) the rotation of the field $x - Fx$ is defined, and Theorems 1-3 are valid. We formulate new ones.

Let F be defined on all of $W\ (K)$ and be equal (cf. with (^{15}))

$$Fx = A(x)x + \varphi(x), \tag{1}$$

$FW \subset W\ (FK \subset K)$, where $A(y)$ is a linear bounded operator in E, depending on the parameter $y \in W\ (K)$ and leaving $W\ (K)$ invariant. Suppose that, for $x, y \in W\ (K)$, the inequalities

$$\inf_{\|x\|=1} \|x - A(y)x\| \geq \gamma^*, \qquad \sup_{\|x\|=\rho} \|\varphi(x)\| < \gamma^* \|x\|, \qquad \gamma^*, \rho > 0. \tag{2}$$

are fulfilled.

Lemma 3. If conditions (1)—(2) are fulfilled and $A(y)x$ is completely continuous in (x, y), then the rotation of the field $x - Fx$ on the sphere $\|x\| = \rho$, $x \in W\ (K)$, is equal to the index $\gamma(\theta)$ of the unique fixed point θ of the linear field $x - A(\theta)x$ in $W\ (K)$.

The index of a linear field in a Banach space has in a number of cases been computed (^{7,17,1}). We compute the index of θ in W. Denote by μ_1 the number equal to 1 if the greatest eigenvalue of the operator A_{11} is less than 1, and equal to 0 otherwise. Below, by $\gamma(A)$ we shall denote the index of the point θ of the operator $A(\theta)$ in $W\ (K)$, and by $\gamma_1(A)$, $\gamma_2(A)$ the indices of θ of the operators A_{11}, A_{22}, respectively, in K_1, $B_2\ (K_2)$.

Lemma 4. Let $A(\theta)$ in E be given by the matrix (A_{ij}), $i, j = 1, 2$, with $A_{12} = 0$, $A(\theta)W \subset W\ (A(\theta)K \subset K)$; suppose that A_{11} does not have 1 as a positive eigenvalue. Then $\gamma(A) = \gamma_1(A) \cdot \gamma_2(A)$.

Lemma 5. Let A_{11} be a U_0-positive operator $^{(13)}$, having a positive eigenvalue $\lambda_0 \neq 1$, and let the cone K_1 be reproducing. Then $\gamma_1(A) = \mu_1$.

Let us note that the index of the operator A_{22} in B_2 is known $^{(7)}$.

Theorem 5. Suppose that the conditions of Lemma 3 are satisfied and $\gamma(A) \neq 0$; then, in the ball $\|x\| < \rho$, $x \in W(K)$, there exists a fixed point of the operator Fx.

If there is another representation of the operator $Fx = B(x)x + \psi(x)$, satisfying the same conditions as (1), but (2) is fulfilled on the sphere $\|x\| = \Delta$, $x \in W(K)$, and if $\gamma(A) \neq \gamma(B)$, then in the layer $\rho < \|x\| < \Delta$ there exists a second fixed point $x \in W(K)$.

Up to now, the strong topology from E has been induced in $W(k)$. Suppose now that in B_1 the weak topology is considered, in B_2 the strong topology, and in $E = B_1 + B_2$ the corresponding Tikhonov topology, inducing a topology in

$W(K)$. A theorem analogous to Theorem 5 holds in the case under consideration if all operators are completely continuous in the topology of E, the unit ball in B_1 is weakly compact, and the cone K_1 admits plastering.

Lemma 4 will not change, and Lemma 5 becomes the assertion:

Let the operator A_{11} have a unique positive eigenvector and a simple positive eigenvalue $\lambda^0 \neq 1$, and let A_{11}^n be strongly compact for some integer $n > 0$. Then $\gamma_1(A) = \mu_1$.

Let us note the topological nature of the theorems: on the one hand, they are a generalization of certain cone theorems of M. A. Krasnosel' skii (13), and on the other hand, of our theorems on weakly continuous operators (1–3) (see also the next item).

Let us also note that Theorems 5-7 can be applied, according to the usual schemes, to prove existence theorems for solutions of systems of nonlinear integral equations and of periodic solutions of systems of differential equations.

3. Let us consider, in a real separable Hilbert space H, a linear bounded self-adjoint operator L, whose positive spectrum is separated from zero, and whose subspace carrying the nonpositive spectrum is finite-dimensional. The inequality $(Lx, x) = r^2,\ r \geq 0$, singles out in H a hyperbolic set H_r (if $r = 0$, it is additionally assumed that L has no more than one simple negative eigenvalue). Introduce in H_r the weak topology $\{U\}$; all sets U may be regarded as bounded. Then, for a weakly continuous field $x - Fx$ acting in H_r, the rotation on the boundaries U is defined and Theorems 1-4 are valid (for $r = 0$, Theorems 1-3). We present a new assertion.

Let $H_- \oplus H_+$ be the decomposition of H corresponding to the spectrum of L. We shall say that a linear operator A has index $k \geq 0$ relative to H_- if H_- is invariant and on it the operator has k eigenvalues greater than 1 (counting their multiplicity—the dimension of the Jordan cell), while 1 is not an eigenvalue on H_-. Suppose that the weakly continuous operator F has two representations

$$Fx = A_i(x)x + \varphi_i(x) \qquad (i = 1, 2),$$

where $A_i(y)$ is a family of linear operators depending on the parameter $y \in H_r$, $A_i(y)x$ is weakly continuous in the totality of the variables (x, y); suppose that H_r is an invariant set for A_i, F; let P_- be the projector onto H_-.

Theorem 6. Let $A_i(y)$ have index k_i relative to H_-, let $A_i(y)H_+ \subset H_+$,

$$\|P_-\varphi_i(x)\| \leq \gamma_i \|P_-x\| \quad \text{for } \|P_-x\| = \rho_i > 0, \quad x \in H_r, \quad \rho_1 < \rho_2, \quad \gamma_i > 0,$$

$$\sup \|(I - A_i(y))^{-1}\|_{H_-} < \gamma_i^{-1}, \quad \|P_-y\| = \rho_i, \quad y \in H_r, \qquad (i = 1, 2).$$

Then on the sphere $\|P\_x\| = \rho_i,\ x \in H_r$, the rotation of the field $x - Fx$ is equal to $(-1)^{k_i}$, and if $k_1 \neq k_2(\mathrm{mod}2)$, then there exist two fixed points of the operator F.

In conclusion the author is pleased to express his gratitude to M. A. Krasnosel' skii and A. S. Shvarts for discussion of certain questions.

Voronezh State
University

Received
21 V 1963

REFERENCES CITED

[1] Yu. G. Borisovich, DAN, **131**, No. 2 (1960).
[2] Yu. G. Borisovich, Tr. Tbilisi Math. Inst., **27** (1960).
[3] Yu. G. Borisovich, DAN, **136**, No. 6 (1961).
[4] Yu. G. Borisovich, Tr. IV All-Union Math. Congress (in press).
[5] N. V. Marchenko, DAN, **147**, No. 5 (1962).
[6] J. Leray, Collection of translations. Mathematics, **4**, 5, 1960.
[7] M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.
[8] M. Nagumo, Am. J. Math., **73** (1951).
[9] E. Rothe, Jowa State College J. Sci., **13** (1939).
[10] F. E. Browder, Duke Math. J., **24**, No. 4 (1957).
[11] F. E. Browder, Duke Math. J., **26**, No. 2 (1959).
[12] J. Leray, J. de Math., **24** (1945).
[13] M. A. Krasnosel' skii, *Positive Solutions of Operator Equations*, Moscow, 1962.
[14] M. M. Day, *Normed Linear Spaces*, IL, 1961.
[15] A. I. Perov, DAN, **124**, No. 4 (1950).
[16] W. T. Kyner, Proc. Am. Math. Soc., **7**, No. 6 (1956).
[17] P. P. Zabreiko, M. A. Krasnosel' skii, DAN, **141**, No. 2 (1961).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.