SOME THEOREMS OF THE THEORY OF Ψ-STABILITY IN COOPERATIVE GAMES

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Abstract

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SOME THEOREMS OF THE THEORY OF Ψ-STABILITY IN COOPERATIVE GAMES

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Consider a cooperative game given by a characteristic function in 0-1-reduced form (see (1)). Consider the set of imputations

$$A = \left\{ a = (a_1, \dots, a_n) : \quad a_i \geqslant 0, \quad \sum_{i=1}^{n} a_i = 1 \right\}.$$

To each coalition $S \subset I_n$ assign the vector $S = \{s_1, \dots, s_n\}$, where $s_i = 1$ if $i \in S$, and $s_i = 0$ if $i \notin S$.

We shall call a system of numbers $(\lambda_1, \dots, \lambda_m)$, $\lambda_j \geqslant 0$, a q-θ-covering of I_n if

$$\sum_{j=1}^{m} \lambda_j S_j = I_n, \quad S_j \subset I_n,$$

where q is the number of positive λ_j, and θ is the set of corresponding coalitions S_j (see (3)). The extreme points of the set of coverings are called reduced coverings; their number is finite. A subset U of the set A is called a core if, for any $\alpha \in U$, the condition

$$\sum_{i \in S} \alpha_i = S \cdot \alpha \geqslant v(S) \qquad \text{for all } S \subset I_n$$

is satisfied.

Any partition τ of the set I_n into disjoint coalitions is called a coalition structure.

Suppose that for each τ a mapping $\Psi(\tau)$ into the set of all coalitions $S \subset I_n$ is given, with $\tau \subset \Psi(\tau)$. A pair $[\alpha, \tau]$ $(\alpha \in A)$ is called Ψ-stable if the following conditions are satisfied: 1) $S \cdot \alpha \geqslant v(S)$ for all $S \in \Psi(\tau)$; 2) if $\alpha_i = 0$ $(= v(\{i\}))$, then $\{i\} \in \tau$.

The concept of Ψ-stability was introduced by Luce (see, for example, (1,4)), as was the concept of k-stability. The known theorems on k-stability concerning classes of symmetric games and games with a quota also belong to him.

In the present paper Ψ-stability is studied by means of methods of linear programming.

Lemma. In order that a system of inequalities of the form

$$S \cdot \alpha \geqslant v(S), \quad S \in \Xi,$$

$$I_n \cdot \alpha = 1$$

(Ξ is some set of coalitions) have a solution, it is necessary and sufficient that, for every reduced q-θ-covering $(\lambda_1, \dots, \lambda_m)$ such that $\theta \subset \Xi$, the inequality

$$\sum_{j=1}^{m} \lambda_j v(S_j) \leqslant 1,$$

be satisfied.

moreover, in order that at least one of the inequalities be strict, it is necessary that

$$\sum_{j=1}^{m} \lambda_j v(S_j) < 1$$

for coverings in which the λ_j corresponding to this inequality is > 0.

The proof is based on the theorems on the solvability of systems of linear inequalities from (2).

Theorem 1. In order that, for some τ, there exist a Ψ-stable pair $[a, \tau]$, it is necessary and sufficient that, for every reduced q-θ-covering $(\lambda_1, \dots, \lambda_m)$, for which $\theta \subset \{\Psi(\tau), \{1\}, \dots, \{n\}\}$, the inequality

$$\sum_{j=1}^{m} \lambda_j v(S_j) \leqslant 1$$

hold, and the inequality must be strict for coverings containing one-element sets not belonging to τ.

Proof. In order that the pair $[a, \tau]$ be Ψ-stable, it is necessary and sufficient that a be a solution of the system of inequalities

$$\begin{aligned}
&S \cdot a \geqslant v(S), \quad S \in \Psi(\tau) && \text{(condition 1)}, \\
&a_i > 0, \quad i \in S \in \tau \text{ and } |S| \geqslant 2 && \text{(condition 2)}, \\
&a_i \geqslant 0 \quad \text{for the remaining } i, && \\
&\mathbf{I}_n \cdot a = 1 &&
\end{aligned} \qquad \{ \; a \in A,$$

and it remains for us only to refer to the lemma just formulated, which completes the proof.

Denote by $\bar{\Gamma}$ a game in which there is no such q-θ-covering $(\lambda_1, \dots, \lambda_m)$ that

$$\sum_{j=1}^{m} \lambda_j v(S_j) \leqslant 1.$$

Corollary 1. Whatever the mapping $\Psi(\tau)$ may be, in the game $\bar{\Gamma}$ there exist no Ψ-stable pairs.

Corollary 2. For a game with a superadditive payoff function there exist Ψ-stable pairs for a certain choice of Ψ.

The assertion is true if, for example, the mapping $\Psi(\tau)$ is a "partition" of the coalitions from τ.

Corollary 3. Suppose that in a game

$$(\lambda_1^{(1)}, \dots, \lambda_m^{(1)}), \dots, (\lambda_1^{(t)}, \dots, \lambda_m^{(t)})$$

are all the reduced q_i-θ_i-coverings for which

$$\sum_{j=1}^{m} \lambda_j^{(i)} v(S_j) \leqslant 1;$$

then, in order that a Ψ-stable pair $[a, \tau]$ exist, it is necessary that

$$\Psi(\tau) \subset \bigcup_{i=1}^{t} \theta_i.$$

Corollary 4 (Theorem 1 from (3)). In order that the game Γ have a core, it is necessary and sufficient that, for every reduced q-θ-covering $(\lambda_1, \dots, \lambda_m)$, one have

$$\sum_{j=1}^{m} \lambda_j v(S_j) \leqslant 1.$$

The proof follows from the fact that if $\Psi(\tau)$, for every τ, is a mapping onto the set of all coalitions, then every pair $[a, \tau]$, where $a \in U$, is Ψ-stable, and conversely, for every Ψ-stable pair $a \in U$.

Let us now consider a function $\Psi(\tau)$ of a certain special form (see (1), pp. 289-290): we assume that $\Psi(\tau)$ consists of all coalitions T for which there exists an $S \in \tau$ such that

$$|T \setminus S| + |S \setminus T| \leqslant k$$

(the modulus sign for a set denotes the number of its elements), where k is a given integer. Ψ-stability in this case is called K-stability.

Recall that a game is called symmetric if $v(S) = v(|S|)$.

Theorem 2 (Theorem 1 from ([4])). *In order for a symmetric game to be k-stable, it is necessary and sufficient that the condition*

$$v(S) \leqslant \frac{|S|}{n} \quad \text{for} \quad |S| = 2, \dots, k+1$$

be satisfied.

Proof. Sufficiency. If $v(S)$ satisfies the condition of the theorem, then the pair

$$\left[\left(\frac{1}{n}, \dots, \frac{1}{n}\right), (\{1\}, \dots, \{n\})\right]$$

is k-stable (see the definition).

Necessity. Let $[a, \tau]$ be k-stable, and let $\tau = (S_1, \dots, S_l)$. Fix some r: $2 \leqslant r \leqslant k + 1$. Consider

$$rI_n = \{1, \dots, n,\ 1, \dots, n, \dots, 1, \dots, n\}.$$

Redistribute the players in rI_n so that a partition of rI_n into sets $S_1', \dots, S_m'$ is obtained such that $|S_j'| = a_{jr}$ (a_j an integer). This operation can be carried out so that to each set from τ there will be added (or subtracted) the least negative (positive) remainder upon division by r. Since $r \leqslant k + 1$, the remainders will not exceed k, and this means that the sets $S_1', \dots, S_m' \in \Psi(\tau)$. They form an m-θ-covering, since

$$\sum_{j=1}^{m} S_j' = rI_n \quad \text{or} \quad \sum_{j=1}^{m} \frac{1}{r} S_j' = I_n.$$

By Theorem 1, it is necessary that

$$\sum_{j=1}^{m} \frac{1}{r} v(S_j') \leqslant 1.$$

Since $|S_j'| = a_{jr}$, considering $v(S)$ superadditive, we obtain

$$v(S_j') \geqslant a_j v(r), \quad j = 1, \dots, m.$$

Thus,

$$1 \geqslant \sum_{j=1}^{m} \frac{1}{r} v(S_j') \geqslant \frac{v(r)}{r} \sum_{j=1}^{m} a_j = v(r) \frac{n}{r},$$

i.e. $v(r) \leqslant r/n$ for any $r = 2, \dots, k+1$, as was required to prove.

Corollary. *In order for a symmetric game to have a core, it is necessary and sufficient that* $v(S) \leqslant \frac{|S|}{n}$ *for every* $S \subset I_n$.

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Note: Figure translations are in progress. See original paper for figures.