# E. K. Isakova

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**Abstract**

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# E. K. Isakova

## A General Boundary-Value Problem for Parabolic Equations in the Plane

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In the domain $Q$ $(0 \le t \le T,\ \gamma_0(t) \le x \le \gamma_1(t))$, consider the linear parabolic equation in the sense of Petrovskii

$$L\left(x, t, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) u(x,t) \equiv$$

$$\equiv \frac{\partial u}{\partial t} + (-1)^p A_{2p}(x,t)\frac{\partial^{2p} u}{\partial x^{2p}} + \sum_{k=0}^{2p-1} A_k(x,t)\frac{\partial^k u}{\partial x^k} = 0, \tag{1}$$

$$A_{2p}(x,t) > 0 \quad \text{for } (x,t) \in Q,$$

where $x = \gamma_r(t)$, $r = 0, 1$, are sufficiently smooth curves in the plane $(x,t)$, having a finite number of points $M_i(x_i, t_i)$, $i = 1, \dots, M$, at which the tangent is parallel to the $x$-axis. We shall moreover assume that, in a neighborhood of the points $M_i$, the equations of these curves have the form

$$|x - x_i| = |t - t_i|^{1/2p+\delta}\psi_i(t),$$

where $\delta > 0$, and $\psi_i(t)$, $i = 1, \dots, M$, have bounded derivatives.

I. For equation (1) we shall consider the general boundary-value problem, which consists in the following: in the domain $Q$ one must find a solution of equation (1) satisfying the conditions

$$u(x,t)\big|_{t=0} = 0, \tag{2}$$

$$B_i^r\left(x, t, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)_{x=\gamma_r(t)} = \varphi_{ri}(t), \quad i = 1, \dots, p,\ r = 0, 1, \tag{3}$$

where $B_i^r$, $i = 1, \dots, p$, $r = 0, 1$, are linear differential operators with respect to $\partial/\partial t$ and $\partial/\partial x$.

This problem for the heat equation in the domain $Q(t \ge 0,\ x \ge 0)$ was first studied by A. N. Tikhonov ([1]). In the multidimensional case, a number of authors have dealt with the general boundary-value problem for parabolic equations (and systems) in cylindrical domains. The most complete results on the study of generalized solutions in cylindrical domains are due to M. S. Agranovich and M. I. Vishik ([2]).

In the present note we restrict ourselves to the case of one parabolic equation, although all the results of this note remain valid also for parabolic systems in the plane*.

**Definition 1.** Equation (1) and conditions (2), (3) in $\overline{Q}$ will be called **compatible in the natural manner** if there exists a function $v(x,t)$ satisfying conditions (2), (3) and such that the functions

$$\frac{\partial^{j+k}}{\partial x^j \partial t^k} L\left(x, t, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) v(x,t)$$

are continuous in $\overline{Q}$ for all $0 \le j \le N-2,\ 0 \le k \le K-1$, where $N$ and $K$ are the orders with respect to $x$ and $t$ of the differential operator in (3).

* In the note the case of homogeneous equations (1) and condition (2) is considered, since the nonhomogeneous case is easily reduced to this one.

Everywhere below it will be assumed that: a) all coefficients in (1), (3) (and also in (11)) are sufficiently smooth; b) equation (1) and conditions (2)–(3) are compatible in the natural way. Therefore conditions (3) may be rewritten in the equivalent form

$$B_i^r u\big|_{x=\gamma_r(t)} \equiv \sum_{k=0}^{2p-1} P_{ik}^r\left(s_r, \frac{\partial}{\partial s_r}\right) \frac{\partial^k u}{\partial x^k}\bigg|_{x=\gamma_r(t)} = \varphi_{ri}(t), \tag{4}$$

$i = 1, \dots, p,\ r = 0, 1$, where $s_r$ is the length of the arc of the curve $x = \gamma_r(t)$, measured from the point $(\gamma_r(0), 0)$, $r = 0, 1$.

**Definition 2.** By the **formal degree** in $\lambda$ of the minor of the matrix

$$\Lambda_r = \|P_{i,2p-1}^r(s_r, \lambda), \dots, P_{i0}^r(s_r, \lambda)\|$$

we shall mean its highest possible degree in $\lambda$, under an arbitrary variation of the coefficients in (3) that are not identically equal to zero.

Let $\beta_r$ denote the maximum formal degree among all minors of the matrix $\Lambda_r$, and let

$$\Lambda_{rp}(\nu_{r1}, \dots, \nu_{rp}) \equiv \|P_{i\nu_{r1}}^r, \dots, P_{i\nu_{rp}}^r\|$$

denote the leftmost minor of order $p$ in the matrix $\Lambda_r$ having formal degree $\beta_r$ $(r = 0, 1)$.

**Definition 3.** Conditions (3) ((4)) will be called **nondegenerate** if the degrees of $\Lambda_{0p}$ and $\Lambda_{1p}$, for all $s_0 \ge 0,\ s_1 \ge 0$, coincide respectively with $\beta_0$ and $\beta_1$.

Let us now consider boundary conditions of the form

$$B_i^r u\big|_{x=\gamma_r(t)} \equiv \left\{ \sum b_{rij}(x,t) \frac{\partial^j u}{\partial x^j} + K_{ri}\left(t, u, \dots, \frac{\partial^{2p-1} u}{\partial x^{2p-1}}\right) \right\}\Bigg|_{x=\gamma_r(t)}, \qquad (5)$$

$$j = 0, 1, \dots, m_{ri}, \qquad 2p-1 \geq m_{rp} > \dots > m_{r1} \geq 0 \quad (m_{rp} \equiv m_r),$$

$$i = 1, \dots, p, \qquad r = 0, 1,$$

where $K_{ri}$ $(i = 1, \dots, p,\ r = 0, 1)$ are linear Volterra-type integral operators with respect to their arguments, with measurable bounded kernels.

**Definition 4.** The boundary conditions (5) will be called **canonical** if $b_{rim_{ri}} \neq 0$ for all $s_r \geq 0,\ i = 1, \dots, p,\ r = 0, 1$.

**Lemma 1.** *If conditions (5) are canonical, and the functions $\varphi_{ri}(t)$ belong to*

$$C^{\chi_i}([0,T]), \qquad \chi_i = 1+\varepsilon-\frac{m_{ri}+p-m_r}{2p}, \qquad \varepsilon > 0, \quad j, i = 1, \dots, p, \quad r = 0, 1,$$

*then, for every $\varepsilon > 0$, problem (1)–(2)–(5) is uniquely solvable in the domain $Q$.*

**Proof.** Using the method of proof of the main theorem in (3), one can show the validity of Lemma 1 in the case $m_0 = m_1 = p - 1$. Therefore, without loss of generality, it may always be assumed that $\varphi_{ri}(t) = 0$ for all $0 \leq m_{ri} \leq p-1,\ i = 1, \dots, p,\ r = 0, 1$.

Let $G_0(x, \xi, t, \tau)$ denote the fundamental solution of equation (1), and let $G_{rk}(x, \xi, t, \tau)$, $r = 0, 1$, denote the functions

$$G_{0k}(x,\xi,t,\tau) = \int_{-\infty}^{\xi=\xi_k} d\xi_{k-1} \cdots \int_{-\infty}^{\xi_1} G_0(x, \xi_0, t, \tau)\, d\xi_0,$$

$$G_{1k}(x,\xi,t,\tau) = \int_{\xi=\xi_k}^{\infty} d\xi_{k-1} \cdots \int_{\xi_1}^{\infty} G_0(x, \xi_0, t, \tau)\, d\xi_0.$$

We shall call a **potential of the $k$-th kind with continuous density** $\mu_{rk}(t)$, **concentrated on the curve** $x = \gamma_r(t)$, the function

$$V_k^r(x,t) = \int_0^t \mu_{rk}(\tau) \frac{\partial^k G_{m_r-p+1}(x,\xi,t,\tau)}{\partial \xi^k}\bigg|_{\xi=\gamma_r(\tau)} d\tau,$$

$$k = 0, 1, \dots, m_r + p - 1.$$

We shall seek the solution of problem (1)–(2)–(5) in the form

$$u(x,t) = V_0^0(x,t) + V_0^1(x,t) + \dots + V_{p-1}^0(x,t) + V_{p-1}^1(x,t). \quad (6)$$

Hence, using (5) and the properties of the fundamental solution $G_0(x,\xi,t,\tau)$, by analogy with (3) we obtain

$$\sum_{j=1}^{m_{ri}} b_{rij} \sum_{k=0}^{p-1} \alpha_{ijk}^r \, \mathcal{I}_{\left(1-\frac{k+i+p-m_r}{2p}\right)}(\mu_{rk}) = \varphi_{ri} + \Phi_{ri}(\mu_{10},\dots,\mu_{0p-1},\mu_{10},\dots,\mu_{1p-1}), \quad (7)$$

where $\alpha_{ijk}^r$ are certain constants, $\Phi_{ri}$ are completely continuous Volterra-type operators, and $\mathcal{I}_\sigma(\mu)$, $\sigma > 0$, is the operator of fractional integration in the Riemann-Liouville sense. (For the definition of fractional integration and fractional differentiation operators see, for example, (8, 4).)

Applying to the $i$-th equation (7) the fractional integration operator of order $m_{ri}/2p$, we obtain

$$\sum_{k=0}^{p-1} \overline{\alpha}_{kri} \, \mathcal{I}_{\left(1-\frac{p+k-m_r}{2p}\right)}(\mu_{rk}) = \mathcal{I}_{\frac{m_{ri}}{2p}}(\varphi_{ri}) + \cdots \quad (8)$$

Equation (8) may be regarded as a linear algebraic system with respect to $\mathcal{I}_{\left(1-\frac{p+k-m_r}{2p}\right)}(\mu_{rk})$, $k = 0,\dots,p-1$, $r = 0,1$. The determinant of this system is not zero, which follows from the unique solvability (see (5)) of problem (1)–(2)–(5) for $\gamma_0(t) \equiv 0$, $\gamma_1(t) \equiv 1$, $K_{0i} \equiv K_{1r} \equiv 0$, $i = 1,\dots,p$. From (8) we find

$$\mathcal{I}_{\left(1-\frac{p+k-m_r}{2p}\right)}(\mu_{rk}) = L_{rk}\Big(\mathcal{I}_{\frac{m_{01}}{2p}}(\varphi_{01}),\dots,\mathcal{I}_{\frac{m_{0p}}{2p}}(\varphi_{0p}),$$

$$\mathcal{I}_{\frac{m_{11}}{2p}}(\varphi_{11}),\dots,\mathcal{I}_{\frac{m_{1p}}{2p}}(\varphi_{1p}),\mu_{10},\dots,\mu_{0p-1},\mu_{01},\dots,\mu_{1p-1}\Big), \quad (9)$$

$k = 0,\dots,p-1$, $r = 0,1$, where the $L_{rk}$ are linear in all their arguments, and with respect to $\mu_{r0},\dots,\mu_{rp-1}$ are also completely continuous and of Volterra type. Applying to the $k$-th equation (9) the fractional differentiation operator

$$D^{1-\frac{p+k-m_r}{2p}},$$

we obtain, with respect to $\mu_{r0},\dots,\mu_{rp-1}$, $r = 0,1$, a system of integral equations of the second kind of Volterra type, which is uniquely solvable. Here, by the operator $D^\alpha$ in the case $\alpha \geq 1$ we understand the successive application first of the operator $d^{[\alpha]}/dt^{[\alpha]}$, and then of the operator $D^{(\alpha)}$ ($\alpha = \alpha - [\alpha]$), as was required to prove.

**Lemma 2.** *The nondegenerate boundary conditions (3), ((4)) can always be reduced to the canonical form (5).*

Indeed, consider conditions (4) with respect to $\omega_{rj} \equiv \partial^{\nu_{rj}} u / \partial x^{\nu_{rj}}\big|_{x=\gamma_r}$, $j = 1, \dots, p$, as a system of ordinary differential equations in $s_r$, $r = 0, 1$. In view of the assumptions made on the compatibility of (1), (2), (3),

$$\left.\frac{\partial^{q_r} \omega_{rj}}{\partial s_r^{q_r}}\right|_{s_r=0} = 0, \qquad q_r = 0, 1, \dots, \eta_{rj}, \quad j = 1, \dots, p, \quad r = 0, 1, \tag{10}$$

where $\eta_{rj}$ is the greatest order of differentiation of the function $\omega_{rj}$ in (4). Solving, for $\omega_{rj}$, $j = 1, \dots, p$, $r = 0, 1$, the Cauchy problem (4)–(10), we obtain the validity of Lemma 2.

With the aid of Lemmas 1 and 2 one proves

**Theorem 1.** *If conditions (3) are nondegenerate, and the functions $\varphi_{ri}(t)$ belong to $C^{\varkappa_r}([0, T])$, where*

$$\varkappa_r = \left(1 + \varepsilon - \frac{p - m_r + m_{r1}}{2p}\right), \quad \varepsilon > 0, \quad r = 0, 1,$$

*$i = 1, \dots, p$, then for any $\varepsilon > 0$ the problem (1)–(2)–(3) is uniquely solvable.*

II. Instead of condition (3), consider the more general boundary conditions

$$B_i^0\left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u\bigg|_{x=\gamma_0(t)} + B_i^1\left(x, t, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)\bigg|_{x=\gamma_1(t)} = \varphi_i(t), \tag{11}$$

$$i = 1, \dots, 2p.$$

We shall continue to assume that equation (1) and conditions (2), (11) are naturally compatible. Then (11) can be rewritten in the form

$$\sum_{k=0}^{2p-1} P_{ik}^0\left(s_0, \frac{\partial}{\partial s_0}\right) \frac{\partial^k u}{\partial x^k}\bigg|_{x=\gamma_0(t)} + \sum_{k=0}^{2p-1} P_{ik}^1\left(s_1, \frac{\partial}{\partial s_1}\right) \frac{\partial^k u}{\partial x^k}\bigg|_{x=\gamma_1} = \varphi_i, \tag{12}$$

$$i = 1, \dots, 2p.$$

In the matrix

$$\Lambda \equiv \left\| P^0_{i\,2p-1}, \dots, P^0_{i0},\ P^1_{i\,2p-1}, \dots, P^1_{i0} \right\|$$

consider minors of order $2p$ of the form

$$\left\| P^0_{i\nu_1}, \dots, P^0_{i\nu_p},\ P^1_{i\sigma_1}, \dots, P^1_{i\sigma_p} \right\| = \Lambda_{2p}(\nu_1, \dots, \nu_p, \sigma_1, \dots, \sigma_p),$$

where

$$0 \le \nu_p < \dots < \nu_1 \le 2p-1, \quad 0 \le \sigma_p < \dots < \sigma_1 \le 2p-1.$$

Let $\beta$ be the greatest formal degree of all such minors, and let

$$\widetilde{\Lambda}_{2p}(\tilde{\nu}_1, \dots, \tilde{\nu}_p, \tilde{\sigma}_1, \dots, \tilde{\sigma}_p)$$

be the leftmost minor among all minors

$$\Lambda_{2p}(\nu_1, \dots, \nu_p, \sigma_1, \dots, \sigma_p)$$

having formal degree $\beta$.

**Definition 5.** Conditions (12) ((11)) will be called **nondegenerate** if the degree of the minor $\widetilde{\Lambda}_{2p}$ is equal to $\beta$.

**Theorem 2.** *If conditions (11) ((12)) are regular uniformly in $t \in [0, T]$, in the sense of* (6), *and nondegenerate, and the functions $\varphi_i(t)$ in (12) belong to $C^{\varkappa}([0, T])$ for some $\varkappa > 0$, then the problem (1)–(2)–(11) is uniquely solvable ($\varkappa$ is computed analogously to $\varkappa_r$ in Theorem 1).*

The proof of this theorem is carried out according to the same plan as that of Theorem 1.

In this case, canonical boundary conditions will be the nondegenerate conditions, regular uniformly in $t \in [0, T]$ in the sense of (6), of the form

$$\sum_{r=0}^{1} \sum_{k=0}^{m_{ri}} b_{rik}(t) \frac{\partial^k u}{\partial x^k}\bigg|_{x=\gamma_r(t)} + K_i \left( t,\, u|_{x=\gamma_0}, \dots, \frac{\partial^{2p-1} u}{\partial x^{2p-1}}\bigg|_{x=\gamma_0}, \right.$$

$$\left. u|_{x=\gamma_1}, \dots, \frac{\partial^{2p-1} u}{\partial x^{2p-1}}\bigg|_{x=\gamma_1} \right), \quad i = 1, \dots, 2p,$$

where $K_i$ are linear integral operators of Volterra type with respect to their arguments, with measurable bounded kernels.

The lemma corresponding to this case is proved by the method of potentials analogously to item I, using at the appropriate place the results of (7).

Computing Center
Academy of Sciences of the USSR

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## REFERENCES

1. A. N. Tikhonov, *UMN*, **1**, 3-4 (1946).

2. M. S. Agranovich, M. I. Vishik, *UMN*, **18**, no. 1, 206 (1963).

3. V. P. Mikhailov, *DAN*, **132**, no. 2 (1960).

4. V. P. Mikhailov, *DAN*, **129**, no. 6 (1959).

5. V. P. Mikhailov, *DAN*, **126**, no. 6 (1959).

6. M. A. Naimark, *Linear Differential Operators*, Moscow, 1954.

7. V. P. Mikhailov, *DAN*, **144**, no. 2 (1962).

8. A. Zygmund, *Trigonometric Series*, IL, 1939.

*Note: Figure translations are in progress. See original paper for figures.*