ON A PROBLEM OF S. N. BERNSTEIN

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Abstract

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Denote by $C_{2\pi}$ the space of continuous functions $x(t)$ defined on $[0, 2\pi]$ and satisfying the condition $x(0) = x(2\pi)$; by $\mathcal{P}_{n,h}(\alpha, \beta)$ the class of trigonometric polynomials $P_n(t)$,

$$P_n(t) = \sum_{k=0}^{n} (a_k \cos kt + b_k \sin kt)$$

of order not exceeding n, with real coefficients a_k and b_k and fixed $a_h = \alpha$ and $b_h = \beta$, $0 < h \leq n$; and by $\mathcal{P}_n$ the class of trigonometric polynomials $P_n(t)$ subject to the condition

$$\sup_{[0,2\pi]} |P_n(t)| = 1.$$

In (1), pp. 28-31, the problem was posed of finding

$$M_{n,h}(\alpha, \beta) = \inf_{P_n(t) \in \mathcal{P}_{n,h}(\alpha,\beta)} \sup_{[0,2\pi]} |P_n(t)|.$$

There S. N. Bernstein obtained an estimate of this quantity and investigated its asymptotics.

In the present article the exact value of $M_{n,h}(\alpha, \beta)$ is found and a polynomial $\pi_{n,h}(\alpha, \beta, t) \in \mathcal{P}_{n,h}(\alpha, \beta)$ is constructed for which

$$\sup_{[0,2\pi]} |\pi_{n,h}(\alpha, \beta, t)| = M_{n,h}(\alpha, \beta).$$

Since $M_{n,h}(\alpha, \beta) = M_{m,1}(\alpha, \beta)$, where $m = E(n/h)$, it is possible to restrict oneself to considering the class $\mathcal{P}_{n,1}(\alpha, \beta)$. Moreover, if the polynomial $\pi_{n,1}(\alpha, \beta, t)$ has the least deviation from zero on $[0, 2\pi]$ in $\mathcal{P}_{n,1}(\alpha, \beta)$, then the polynomial $A\pi_{n,1}(\alpha, \beta, t + \varphi)$, where

$$A^2 = \frac{1}{\alpha^2 + \beta^2}, \qquad \varphi = \operatorname{arc\,tg} \frac{\beta}{\alpha},$$

belongs to $\mathcal{P}_{n,1}(1,0)$, and its deviation from zero is the least in this class. Consequently, it suffices to indicate in $\mathcal{P}_{n,1}(1,0)$ a polynomial with least deviation from zero and to find $M_{n,1}(1,0)$. This last problem may be replaced by the following: in $\mathcal{P}_n$, find a polynomial whose coefficient a_1 is largest, and find

$$\sup_{P_n} a_1.$$

Below precisely this formulation of the problem is considered.

The article uses the method of functionals proposed by E. V. Voronovskaya ([2],[3]).

A linear functional F on $C_{2\pi}$ may be specified by a moment sequence $(\lambda_k)_{-\infty}^{\infty}$,

$$\lambda_k = \int_0^{2\pi} e^{ikt}\, dg(t), \qquad k = 0, \pm 1, \dots,$$

where $g(t)$ is a real-valued function of bounded variation on $[0, 2\pi]$, with

$$\operatorname{Var}_{[0,2\pi]} g(t) = \|F\|.$$

A function $x(t) \in C_{2\pi}$, $\sup_{[0,2\pi]} |x(t)| = 1$, will be called extremal for F if $F(x) = \|F\|$.

The segment $(\lambda_k)_{-n}^{n}$, $\lambda_k = \overline{\lambda}_{-k}$, defines a functional F_n on the set $\mathcal{P}_n$,

$$F_n(P_n) = \sum_{k=-n}^{n} c_k \lambda_k, \qquad P_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}, \qquad P_n(t) \in \mathcal{P}_n.$$

The condition necessary and sufficient for $P_n(t)$, $P_n(t) \in \mathcal{P}_n$, $P_n(t) \not\equiv 1$, to be extremal for F_n, consists in the fulfillment of the equalities

$$P_n(t_j) = \operatorname{sign} \delta_j. \tag{1}$$

for all nonzero δ_j. Here $0 \le t_1 < t_2 < \dots < t_s < 2\pi$ are the deviation points of $P_n(t)$ on $[0, 2\pi)$, and $(\delta_j)_s$ are found from the system

$$\lambda_k = \sum_{j=1}^{s} \theta_j^k \delta_j, \qquad \theta_j = e^{it_j}, \qquad k = 0, \pm 1, \dots, \pm n. \tag{2}$$

In the case when (1) is fulfilled, $\|F_n\| = \sum_{j=1}^{s} |\delta_j|$, and the extension $F_n = (\lambda_k)_{-n}^{n}$ to the sets $\mathcal{P}_{n+p}$, $p = 1, 2, \dots$, with preservation of norms is unique and is realized by the numbers

$$\lambda_{n+p} = \sum_{j=1}^{s} \theta_j^{n+p} \delta_j, \qquad p = 1, 2, \dots .$$

Consider the functional $F_{n,h} = (\lambda_k)_{-n}^{n}$, where $\lambda_k = 0$, $k = 0, 1, \dots, h-1, h+1, \dots, n$; $\lambda_n = 1$; $\lambda_k = \lambda_{-k}$. Obviously, $F_{n,h}(P_n) = c_{-h} + c_h = a_h$. Thus, the problem is reduced to finding an extremal polynomial and the norm of the functional $F_{n,1}$.

Lemma. Let

$$\sigma_k = \begin{cases} \cos\dfrac{2k-1}{2(l+2)}\pi, & k = 1, 2, \dots, \frac{1}{2}(l+1), \\ \cos\dfrac{2k+1}{2(l+2)}\pi, & k = \frac{1}{2}(l+3), \dots, l+1, \end{cases} \qquad \text{for } l \text{ odd;}$$

$$\sigma_k = \begin{cases} \cos\dfrac{k-1}{l+2}\pi, & k = 1, 2, \dots, \frac{1}{2}l+1, \\ \cos\dfrac{k}{l+2}\pi, & k = \frac{1}{2}l+2, \dots, l+2, \end{cases} \qquad \text{for } l \text{ even.}$$

There exists an algebraic polynomial $H_l(x)$ of degree $2l+1$ such that

$$\sup_{[-1,1]} |H_l(x)| = 1;$$

$$H_l(\sigma_k) = \begin{cases} +1, & k = 1, 2, \dots, \frac{1}{2}(l+1), \\ -1, & k = \frac{1}{2}(l+3), \dots, l+1, \end{cases} \qquad \text{for } l \text{ odd;}$$

$$H_l(\sigma_k) = \begin{cases} +1, & k = 1, 2, \dots, \frac{1}{2}l+1, \\ -1, & k = \frac{1}{2}l+2, \dots, l+2, \end{cases} \qquad \text{for } l \text{ even.}$$

Proof. The polynomial $H_l(x)$ is the Hermite interpolation polynomial constructed at the nodes σ_k:

$$H_l(x) = 2\,\frac{T_{l+2}^2(x)}{(l+2)^2 x} \sum_{k=1}^{\frac{1}{2}(l+1)} \sigma_k \frac{\sigma_k^4 - (3\sigma_k^2 - 2)x^2}{(x^2 - \sigma_k^2)^2} \qquad \text{for } l \text{ odd,}$$

$$H_l(x) = \frac{T_{l+2}^2(x)}{(l+2)^4 x} \left[1 - 2(x^2 - 1) \sum_{k=2}^{\frac{1}{2}l+1} \sigma_k \frac{\sigma_k^4 - (3\sigma_k^2 - 2)x^2}{(x^2 - \sigma_k^2)^2} \right] \qquad \text{for } l \text{ even.}$$

Here $T_{l+2}(x) = \cos(l+2)\arccos x$. The condition $\sup_{[-1,1]} |H_l(x)| = 1$ is easily verified if one observes that the expression $\sigma_k^4 - (3\sigma_k^2 - 2)x^2$ is nonnegative on $[-1, 1]$ for all k.

Theorem. *Let* $2l+1 \leq n < 2l+3,\ l = 0, 1, 2, \ldots$. *The extremal polynomial for* $F_{n,1}$ *is* $H_l(\mathrm{const})$.

Proof. We write the system (2) for the functional $F_{2l+1,1}$ and the polynomial $H_l(\cos t)$. In our case $s = 2(l+1)$, and the points $(\theta_j)_1^{2(l+1)}$ are the roots of the polynomial

$$\frac{z^{2(l+2)} - (-1)^l}{z^2+1}.$$

From (2) we find

$$\delta_k = (-1)^k \frac{\theta_k^{2(l+1)}}{\prod\limits_{j=1}^{k-1}(\theta_k - \theta_j) \prod\limits_{j=k+1}^{2(l+1)} (\theta_j - \theta_k)} \qquad k = 1, 2, \ldots, 2(l+1);$$

$$\arg \delta_k = (k + 1/2)\pi + (l+2)t_k \qquad \text{for } l \text{ odd},$$

$$\arg \delta_k = (k+1)\pi + (l+2)t_k \qquad \text{for } l \text{ even}.$$

It is now easy to establish the equalities (1) for the found δ_k. Thus the theorem is proved for $n = 2l+1$. Extending $F_{2l+1,1}$ to the set $\mathcal{P}_{2l+2}$ with preservation of the norm, we obtain

$$\lambda_{2l+2} = \sum_{j=1}^{2(l+1)} \theta_j^{2l+2} \delta_j = 0,$$

i.e. the indicated extension of the functional $F_{2l+1,1}$ is the functional $F_{2l+2,1}$. Since the polynomial $H_l(\cos t)$ is extremal for $F_{2l+1,1}$, it is also extremal for $F_{2l+2,1}$.

Corollary 1. *If* $2l+1 \leq n < 2l+3,\ l = 0, 1, 2, \ldots,$ *then*

$$\sup_{\mathcal{P}_n} a_1 = \frac{2}{l+2} \operatorname{ctg} \frac{\pi}{2(l+2)}.$$

Indeed, for the indicated n,

$$\sup_{\mathcal{P}_n} a_1 = \sup_{\mathcal{P}_{2l+1}} a_1 = \|F_{2l+1,1}\| = \sum_{k=1}^{2(l+1)} |\delta_k| = \sum_{k=1}^{2(l+1)} \frac{1}{\prod\limits_{\substack{j=1 \\ j \neq k}}^{2(l+1)} |\theta_j - \theta_k|} = \frac{2}{l+2} \operatorname{ctg} \frac{1}{2(l+2)}.$$

Corollary 2. *If* $2l+1 \leq n < 2l+3,\ l=0,1,2,\dots,$ *then*

$$M_{n,1}(1,0) = \frac{1}{2}(l+2)\operatorname{tg}\frac{\pi}{2(l+2)}, \qquad \pi_{n,1}(1,0,t) = M_{n,1}(1,0)H_l(\cos t).$$

The polynomial $H_l(\cos ht)$ is, obviously, extremal for $F_{n,h}$ when $(2l+1)h \leq n < (2l+3)h,\ l=0,1,2,\dots,\ 0<h\leq n$. Any other extremal polynomial of this functional among its points of deviation on $[0,2\pi)$ must necessarily contain all points of deviation $(t_i)_1^{2h(l+1)}$ of the polynomial $H_l(\cos ht)$. Hence it follows that

Corollary 3. *If* $(2l+1)h \leq n < (2l+2)h,\ l=0,1,2,\dots,\ 0<h\leq n,$ *then the polynomial* $H_l(\cos ht)$ *is the unique extremal polynomial* $F_{n,h}$. *For* $(2l+2)h \leq n < (2l+3)h$ *any extremal poly-*

the polynomial $P_{n,h}(t)$ of the functional $F_{n,h}$ can be written in the form

$$P_{n,h}(t) = H_l(\cos ht) + \psi(t)\prod_{j=1}^{2h(l+1)}\sin^2\frac{t-t_j}{2},$$

where $\psi(t)$ is a polynomial whose choice is restricted only by the condition $P_{n,h}(t)\in\mathcal{P}_n$.

Corollary 4. If $(2l+1)h \leqslant n < (2l+3)h,\ l=0,1,2,\dots,\ 0<h\leqslant n$, then

$$M_{n,h}(\alpha,\beta) = \frac{1}{2}\sqrt{\alpha^2+\beta^2}\,(l+2)\operatorname{tg}\frac{\pi}{2(l+2)};$$

$$\pi_{n,h}(x,\beta,t) = M_{n,h}(\alpha,\beta)\,H_l[\cos(ht-\varphi)], \qquad \varphi = \operatorname{arc\,tg}\frac{\beta}{\alpha},$$

where $\pi_{n,h}(\alpha,\beta,t)$ is the unique polynomial least deviating from zero on $[0,2\pi]$ in the class $\mathcal{P}_{n,h}(\alpha,\beta)$, when $(2k+1)h \leqslant n < (2l+2)h$.

The general form of the polynomial of least deviation for the remaining n is clear from the preceding corollary.

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