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Abstract

Full Text

MATHEMATICS

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CONJUGATE FUNCTIONS OF TWO VARIABLES AND DOUBLE FOURIER SERIES

(Presented by Academician A. N. Kolmogorov, 20 X 1962)

1. It is known (see ⁽⁴⁾)* that for every integrable function $f(x)$ there exists almost everywhere the integral

$$\bar{f}(x) = -\frac{1}{2\pi} \int_0^\pi [f(x+t) - f(x-t)] \operatorname{ctg} \frac{t}{2} dt,$$

called the function conjugate to $f(x)$. Riesz ⁽⁷⁾ established that if $f(x) \in L^p(-\pi, \pi)$, $p > 1$, then $\bar{f}(x) \in L^p(-\pi, \pi)$ and $\|\bar{f}\|_{L^p} \leq A(p)\|f\|_{L^p}$. If $p = 1$, then Riesz's theorem does not hold, since $\bar{f}(x)$ may fail to be summable (⁽³⁾, p. 227). But the following theorem of A. N. Kolmogorov is valid:

If $f(x)$ is integrable, then $|\bar{f}(x)|^p$ is integrable, and

$$\left\{ \int_{-\pi}^{\pi} |\bar{f}(x)|^p dx \right\}^{1/p} \leq B(p) \int_{-\pi}^{\pi} |f(x)| dx, \quad 0 < p < 1,$$

where $B(p)$ is a constant depending only on p ⁽²⁾.

A. Zygmund observed ⁽⁹⁾ that if $|f(x)| \log^+ |f(x)| \in L(-\pi, \pi)$, then $\bar{f}(x)$ is summable and

$$\int_{-\pi}^{\pi} |\bar{f}(x)| dx \leq A \int_{-\pi}^{\pi} |f(x)| \log^+ |f(x)| dx + B,$$

where A and B are constants.

In the case when $f(x) \geq 0$, this theorem admits a converse, i.e., if $f(x) \in L(-\pi, \pi)$, $f(x) \geq 0$, and $\bar{f}(x)$ is summable, then $|f(x)| \log^+ |f(x)|$ is summable ⁽⁷⁾.

Let now $f(x, y)$, 2π -periodic with respect to each of the variables, be summable on $R_0 = [-\pi, \pi; -\pi, \pi]$. Consider the functions

$$\begin{aligned}\bar{f}_1(x, y) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+s, y) \operatorname{ctg} \frac{s}{2} ds, \\ \bar{f}_2(x, y) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y+t) \operatorname{ctg} \frac{t}{2} dt, \\ \bar{f}(x, y) &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+s, y+t) \operatorname{ctg} \frac{s}{2} \operatorname{ctg} \frac{t}{2} ds dt.\end{aligned}$$

Following Cesari ⁽⁸⁾, we shall call the functions $\bar{f}_1(x, y)$, $\bar{f}_2(x, y)$, $\bar{f}(x, y)$ conjugate respectively with respect to the variable x , with respect to the variable y , and

* For $f(x) \in L^2(-\pi, \pi)$ this result was earlier obtained by N. N. Luzin ⁽³⁾, p. 217).

with respect to the set of variables x and y . Sokol-Sokolovskii ⁽⁹⁾ established that the Riesz theorem is also valid for conjugate functions of two variables. If, however, $|f(x, y)| \log^+ |f(x, y)| \in L(R_0)$, then, as Zygmund ⁽¹⁰⁾ proved, the function $\bar{f}(x, y)$ exists almost everywhere and

$$\begin{aligned}& \left\{ \iint_{R_0} |\bar{f}(x, y)|^p dx dy \right\}^{1/p} \leq \\ & \leq A'(p) \iint_{R_0} |f(x, y)| \log^+ |f(x, y)| dx dy + B'(p), \quad 0 < p < 1,\end{aligned}$$

where $A'(p)$ and $B'(p)$ are constants depending only on p .

However, it is unknown whether the condition $|f(x, y)| \log^+ |f(x, y)| \in L(R_0)$ implies that $\bar{f}(x, y) \in L(R_0)$.

2. In this section we shall give a complete answer to the question just posed and present a number of other assertions belonging to the same circle of ideas.

Theorem 1. There exists a function $f(x, y)$, 2π -periodic in each of the variables x and y , such that

$$|f(x, y)| (\log^+ |f(x, y)|)^{2-\varepsilon} \in L(R_0)$$

for every $0 < \varepsilon < 2$, but $\bar{f}(x, y)$ is not summable.

Theorem 2. There exists a nonnegative 2π -periodic function $f(x, y) \in L(R_0)$ for which $\bar{f}_1(x, y)$ is not summable on any interval with respect to y for any fixed $x \in E$, where $|E| > 0$.

Theorem 3. If $|f(x, y)| \log^+ |f(x, y)| \in L(R_0)$, then almost everywhere

$$\bar{f}(x, y) = \bar{f}_{1,2}(x, y) = \bar{f}_{2,1}(x, y).$$

This theorem was published by us in ⁽¹¹⁾.

It is easy to prove that if

$$|f(x, y)|(\log^+ |f(x, y)|)^\alpha \in L(R_0), \quad \alpha > 0,$$

then

$$|\bar{f}_i(x, y)| \log^{\alpha-1}(2 + |\bar{f}_i(x, y)|) \in L(R_0) \quad (i = 1, 2).$$

Consequently, from Theorems 1 and 3 it follows immediately:

Corollary. If

$$|f(x, y)|(\log^+ |f(x, y)|)^2 \in L(R_0), \quad (1)$$

then

$$\bar{f}(x, y) \in L(R_0), \quad (2)$$

and assertion (2) loses its force if, in condition (1), the square is replaced by a smaller power.

An assertion of analogous type also holds for functions of n variables. For example, the following is true.

Theorem 4. If

$$|f(x_1, x_2, \dots, x_n)|(\log^+ |f(x_1, x_2, \dots, x_n)|)^n \in L(R'_0), \quad (3)$$

then

$$\bar{f}(x_1, x_2, \dots, x_n) \in L(R'_0), \quad (4)$$

and assertion (4) loses its force if, in condition (3), the power n is replaced by a smaller one,

$$R'_0 = [-\pi, \pi, -\pi, \pi; \dots; -\pi, \pi].$$

3. In this section we shall give a number of assertions concerning (C, α, β) -summability of double Fourier series and their conjugate series.

Let the series

$$\begin{aligned} \sum_{m,n=0}^{\infty} \lambda_{mn} (a_{mn} \cos mx \cos ny + b_{mn} \sin mx \cos ny + \\ + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \sin ny) \end{aligned} \quad (5)$$

be the double Fourier-Lebesgue series of the function $f(x, y)$, where $\lambda_{00} = 1/4$, $\lambda_{0n} = \lambda_{m0} = \lambda_{mn} = 1$, $m, n > 0$. Denote by $\sigma_{mn}^{\alpha, \beta}(x, y)$ the Cesàro (C, α, β) -means of the series (5). Further, let (cf. (1))

$$\begin{aligned}\Delta_{h\eta}f(x, y) &= f(x + h, y + \eta) - f(x + h, y) - f(x, y + \eta) + f(x, y), \\ \Delta_h f(x, y) &= f(x + h, y) - f(x, y), \quad \Delta_\eta f(x, y) = f(x, y + \eta) - f(x, y).\end{aligned}$$

Theorem 5. If the function $f(x, y)$, continuous and 2π -periodic with respect to each of the variables x and y , satisfies the conditions:

$$\begin{aligned}\left\{ \iint_{R_0} |\Delta_{h\eta}f(x, y)|^{p_1} dx dy \right\}^{1/p_1} &= O(h^\alpha \eta^\beta); \\ \left\{ \int_{-\pi}^{\pi} |\Delta_h f(x, y)|^{p_2} dx \right\}^{1/p_2} &= O(h^{\alpha'}) \quad \text{uniformly with respect to } y; \\ \left\{ \int_{-\pi}^{\pi} |\Delta_\eta f(x, y)|^{p_3} dy \right\}^{1/p_3} &= O(h^{\beta'}) \quad \text{uniformly with respect to } x,\end{aligned}$$

then

$$\|\sigma_{mn}^{-\lambda, -\delta}(x, y) - f(x, y)\|_C = O\left(m^{-\alpha+1/p_1} + m^{-\alpha'+1/p_2} + n^{-\beta+1/p_1} + n^{-\beta'+1/p_3}\right),$$

where

$$0 < \alpha, \beta, \alpha', \beta' < 1, \quad \lambda, \delta < \frac{1}{p_k} \quad (k = 1, 2, 3),$$

$$p_1 > \max\left\{\frac{1}{\alpha}, \frac{1}{\beta}\right\}, \quad p_2 > \frac{1}{\alpha'}, \quad p_3 > \frac{1}{\beta'}.$$

If, however, $\lambda = \frac{1}{p_1}$, then

$$\begin{aligned}\|\sigma_{mn}^{-\lambda, -\delta}(x, y) - f(x, y)\|_C &= O\left(\frac{\ln m^{1-1/p_1}}{m^{\alpha+1/p_1}}\right) \\ &+ m^{-\alpha'+1/p_2} + n^{-\beta+1/p_1} + n^{-\beta'+1/p_3}.\end{aligned}$$

Analogous estimates are obtained if $\lambda = \frac{1}{p_k}$ or $\delta = \frac{1}{p_k}$ ($k = 1, 2, 3$). The theorem remains valid also for double conjugate trigonometric series in the sense of Cesàro ⁽⁸⁾.

Theorem 6. If $f(x, y)$ is a continuous and 2π -periodic function satisfying the condition

$$\left\{ \iint_{R_0} |f(x+h, y+\eta) - f(x, y)|^p dx dy \right\}^{1/p} = O(h^\alpha + \eta^\beta),$$

$$0 < \alpha, \beta < 1, \quad p \geq \max \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\},$$

then the series $\sigma[f]$ is uniformly $(C, -\lambda, -\delta)$ -summable to $f(x, y)$, i.e.

$$\sigma_{mn}^{-\lambda, -\delta}(x, y) - f(x, y) \rightarrow 0, \quad \lambda, \delta < \frac{\min(\alpha, \beta)}{2}.$$

If $p > \max \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\}$, then analogous theorems are valid also for double conjugate trigonometric series.

Theorem 7. If $f(x, y)$ is a continuous 2π -periodic function, then

$$\|\sigma_{mn}^{\alpha, \beta}(x, y) - f(x, y)\|_C = O(\varphi_{mn}),$$

where

$$\varphi_{mn} = \omega_1 \left(\frac{\ln m}{m} \right) + \omega_2 \left(\frac{\ln n}{n} \right), \quad \alpha, \beta > 0,$$

$$\omega_1(\delta) = \max_y \left\{ \sup_{|x_2 - x_1| \leq \delta} |f(x_2, y) - f(x_1, y)| \right\},$$

$$\omega_2(\delta) = \max_x \left\{ \sup_{|y_2 - y_1| \leq \delta} |f(x, y_2) - f(x, y_1)| \right\}.$$

On the basis of results of V. G. Chelidze ⁽⁵⁾, it is proved that under the hypotheses of Theorems 6 and 5 the following relations hold:

$$\|\sigma_{mn}^{\alpha, \beta}(x, y) - f(x, y)\|_C \leq A'' \|\sigma_{mn}^{-\lambda, -\delta}(x, y) - f(x, y)\|_C,$$

$$\alpha > -\lambda, \quad \beta > -\delta.$$

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