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Abstract

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A THEOREM ON THREE BALLS

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In this note Hadamard' s theorem on three circles for analytic functions of a complex variable ([1], p. 469) is generalized to solutions of linear elliptic equations of the second order with many independent variables.

We consider the equation

$$\sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0, \tag{1}$$

whose coefficients, throughout the domain where the equation is defined, satisfy the conditions

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \geq \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \alpha > 0; \tag{2}$$

all coefficients are bounded in absolute value by a constant M; the coefficients a_{ik} are twice continuously differentiable and all their first and second partial derivatives are bounded in absolute value by the constant M; the remaining coefficients are continuously differentiable and their derivatives are bounded by the same constant M; and, finally,

$$c(x) \leq 0. \tag{3}$$

Theorem. *Let in the ball Q of radius $r_2 < 1$ with center at the origin there be defined a solution $u(x)$ of equation (1), continuous in the closed ball. Denote*

$$M(r) = \max_{|x|=r} |u(x)|.$$

Then for any r_1 and r, $0 < r_1 < r < r_2$, the inequality

$$\ln M(r) \leq \ln M(r_1)\frac{\ln Cr/r_2}{\ln r_1/r_2} + \ln M(r_2)\frac{\ln Cr/r_1}{\ln r_2/r_1} + \ln\ln\frac{C}{r}, \tag{4}$$

holds, where C is a constant depending on the constant α in inequality (2), on M, and on the dimension n of the space.

The proof is close in idea to the proofs of the uniqueness theorem for the solution of the Cauchy problem for an elliptic equation given by Heinz ([2]) and Cordes ([3]).

The presence in the right-hand side of inequality (4) of the additional term $\ln\ln\frac{C}{r}$ is possibly connected with the method of proof. Condition (3) is inessential and has been introduced to simplify the proof.

We outline the main steps of the proof.

1°. It is enough to prove the following assertion:

Let in the ball Q_1 of radius 1 there be defined a solution $u(x)$ of equation (1), continuous in the closed ball. Let $M(1) \le 1$, and let, for some r_1, $0 < r_1 < 1$, $M(r_1) = r_1^\beta$, $\beta > 0$.

Then for every r, $r_1 < r < 1$, the inequality

$$M(r) \le (Cr)^\beta \ln\frac{C}{r}, \tag{5}$$

holds, where C is a constant depending on α, M, and n.

2°. From Cordes' results ([3]) it follows that it is enough to prove the assertion of item 1° for the equation

$$Lu \equiv \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}M_r u + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = 0,$$

where M_r, for each fixed r, is a linear elliptic self-adjoint operator on the unit sphere K_1, satisfying the following condition: for every function ω, defined and twice continuously differentiable on the sphere K_1, the inequality

$$\frac{d}{dr}\int_{K_1} \omega M_r \omega \, d\sigma_1 \le 0$$

holds, where $d\sigma_1$ is the surface element of the unit sphere K_1.

3°. Let $f(r)$ be some fixed twice continuously differentiable function on $[1/2,\, 3/4]$, having the properties:

$$0 \le f(r) \le 1, \qquad f(1/2) = 1, \qquad f(3/4) = 0,$$
$$f'(1/2) = f''(1/2) = f'(3/4) = f''(3/4) = 0.$$

Fix some r_0, $r_1 < r_0 < 1$, and put

$$v(x) = \begin{cases} u(x), & \text{for } \frac{1}{2}r_1 \le |x| \le \frac{1}{2}r_0, \\ u(x) f\left(\frac{1}{r_0}|x|\right), & \text{for } \frac{1}{2}r_0 < |x| \le \frac{3}{4}r_0, \\ 0, & \text{for } \frac{3}{4}r_0 < |x| \le r_0. \end{cases}$$

Then for the function $v(x)$, by the maximum principle and Bernstein's inequality ([4]), we have

$$|v|_{|x|=r_1/2} \le r_1^{\beta}; \qquad \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}\right|_{|x|=r_1/2} < C_1 (2r_1)^{\beta-1};$$

$$\sum_{i,k=1}^{n} \left|\frac{\partial^2 v}{\partial x_i \partial x_k}\right|_{|x|=r_1/2} < C_1 (2r_1)^{\beta-2}; \tag{6}$$

$$Lv(x) = 0 \qquad \text{for } \frac{1}{2}r_1 \le |x| \le \frac{1}{2}r_0, \tag{7}$$

$$|Lv(x)| < \frac{C_2}{r_0^2} \qquad \text{for } \frac{1}{2}r_0 < |x| < r_0.$$

The constants C, provided with indices, here and below will denote constants depending only on α, M, and n.

4°. Put

$$\frac{\partial}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2} M_r \equiv L_1.$$

It can be shown, using (6), that

$$\int_{\frac{1}{2}r_1 < |x| < r_0} \frac{v^2}{r^{2\beta+n} \ln^2 \frac{1}{r}}\, dx \le \frac{C_3}{\beta^2} \int_{\frac{1}{2}r_1 < |x| < r_0} \frac{(L_1 v)^2}{r^{2\beta+n-4}}\, dx + C_4 \beta. \tag{8}$$

Further it is shown that

$$\int_{\frac{1}{2}r_1 < |x| < r_0} \frac{|\operatorname{grad} v|^2}{r^{2\beta+n-4}}\, dx \le C_5 r_0 \int_{\frac{1}{2}r_1 < |x| < r_0} \frac{(L_1 v)^2}{r^{2\beta+n-4}}\, dx + C_6 \beta^3. \tag{9}$$

We can choose r_0 so that from (8) and (9) we obtain

$$\int_{\frac{1}{2}r_1 < |x| < r_0} \frac{v^2}{r^{2\beta+n} \ln^2 \frac{1}{r}}\, dx \le \frac{C_7}{\beta^2} \int_{\frac{1}{2}r_1 < |x| < r_0} \frac{(Lv)^2}{r^{2\beta+n-4}}\, dx + C_8 \beta.$$

Here the choice of r_0 depends on the constants C_i, i.e., on α, M, and n. Applying inequality (7), we obtain

$$\int_{\frac{1}{2}r_1<|x|<r_0} \frac{v^2}{r^{2\beta+n}\ln^2\frac{1}{r}}\,dx \le \frac{C_9}{\beta^2 r_0^{2\beta-2}} + C_{10}\beta. \tag{10}$$

5°. Suppose that $\beta > 1$, $r_0 < \frac{1}{2}$, and let r, $\frac{1}{2}r_1 < 2r < r_0$, be an arbitrary number. Then from (10)

$$\frac{1}{(2r)^n}\int_{\frac{1}{2}r_1<|x|<2r} v^2\,dx \le C_{11}\beta\left(\frac{2r}{r_0}\right)^{2\beta}\ln^2\frac{1}{r}.$$

But from this inequality it follows (see (5)) that

$$|v|_{|x|=r} \le C_{12}\left(\frac{2r}{r_0}\right)^{\beta}\ln\frac{1}{r} \le (C_{13}r)^{\beta}\ln\frac{1}{r}$$

or

$$|u|_{|x|=r} \le (C_{13}r)^{\beta}\ln\frac{1}{r} \tag{11}$$

for all r, $r_1 < r < \frac{1}{2}r_0$, and $\beta > 1$.

Since for $\beta \le 1$ the inequality (5) that we need is obtained at once from the fact that $|\operatorname{grad} u|_{|x|<1/2} < C_{15}$, while for $r \ge \frac{1}{2}r_0$ and sufficiently large C_{14} inequality (5) is obvious, it follows that for every r, $r_1 < r < 1$, we have

$$M(r) < (C_{14}r)^{\beta}\ln\frac{1}{r},$$

as was required to prove.

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Note: Figure translations are in progress. See original paper for figures.