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Abstract

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MATHEMATICS

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ON POLYNOMIALS ORTHOGONAL ON A GIVEN SYSTEM OF ARCS OF THE UNIT CIRCLE

(Presented by Academician S. N. Bernstein on 29 I 1963)

1. In the well-known theory of polynomials orthogonal on the circle, created by G. Szegő (1) and developed in the works of Ya. L. Geronimus (2), the case in which the weight vanishes on a set of positive measure is excluded. In the paper of N. I. Akhiezer (3), the general theory is extended to the case where the weight vanishes on an entire arc. The constructions of that paper, however, do not apply if the number of "empty" arcs is greater than one. In comparison with the analogous question for polynomials orthogonal on the line, to which the papers (4,5) are devoted, certain additional difficulties arise here.

2. Let the support of the spectrum be a finite system of arcs

$$(E) \qquad [e^{i\delta}, e^{i\alpha_1}],\ [e^{i\beta_1}, e^{i\alpha_2}], \dots,\ [e^{i\beta_\rho}, e^{i(2\pi-\delta)}],$$

where $0 < \delta < \alpha_1 < \beta_1 < \dots < \alpha_\rho < \beta_\rho < 2\pi - \delta$. Introduce the polynomials

$$S_1(z) = \prod_{k=1}^{\rho} \frac{1}{2i}\left(ze^{-i\alpha_k/2} - e^{i\alpha_k/2}\right); \qquad T(z) = \prod_{k=1}^{\rho} \frac{1}{2i}\left(ze^{-i\beta_k/2} - e^{i\beta_k/2}\right);$$

$$S(z) = \frac{1}{2i}\left(ze^{-i\alpha_0/2} - e^{i\alpha_0/2}\right) S_1(z); \qquad R(z) = S(z)T(z),$$

where $\alpha_0 = 2\pi - \delta,\ \beta_0 = \delta$. Denote by $t(z)$ a continuous and positive function on E, and take as differential weights on E the expressions

$$\sqrt{-\frac{S(z)}{T(z)}}\,\frac{dz}{izt(z)}, \qquad \sqrt{-\frac{T(z)}{S(z)}}\,\frac{dz}{izt(z)}.$$

The corresponding orthogonal polynomials of degree n with leading coefficient equal to unity will be denoted, respectively, by $A_n(z;t)$, $B_n(z;t)$.

Among the weights, those are distinguished (we shall call them special) for which $t(z)$ is a quasipolynomial

$$P(z) = \sum_{k=-\rho}^{\rho} c_k z^k,$$

positive on E.

The investigation is divided into two stages: the construction, for $n \geq 2\rho$, of the orthogonal polynomials $A_n(z;P)$, $B_n(z;P)$ corresponding to a special weight, and the approximation of "arbitrary" weights by special ones and the consequent derivation of asymptotic formulas for the orthogonal polynomials corresponding to arbitrary weights.

The case of a special weight requires, as in (4,5), certain considerations on the two-sheeted Riemann surface $\mathfrak{R}$, whose transition lines are the arcs of the system E. Denoting by z a point on one sheet, and by z' the corresponding point on the other sheet, we establish the following proposition:

Theorem 1. If $p(z, \sqrt{R(z)})$ is a rational function having a pole of order $n \geq 2\rho$ at the point ∞' (i.e., at the point at infinity of the second sheet), $\rho + 1$ simple poles at the points $e^{i\alpha_k}$ $(k = 0, 1, \dots, \rho)$, a pole of order $\rho - 1$ at the point ∞, a zero of multiplicity $n - \rho$ at the point 0, and 2ρ zeros a_k (on the first sheet), where $P(z)$ vanishes, and if

$$\lim_{z \to \infty'} z^{-n} p(z, \sqrt{R(z)}) = 2, \qquad \sigma = \frac{1}{4} \sum_{k=0}^{\rho} (\beta_k - \alpha_k), \tag{1}$$

then

$$p\left(z, \sqrt{R(z)}\right) = A_n(z;P) - e^{i\sigma} \frac{\sqrt{R(z)}}{S(z)} B_n(z;P).$$

It should be noted that $p\left(z, \sqrt{R(z)}\right)$ has another ρ zeros, which are already arbitrary and may lie both on the first and on the second sheet. Denote them by $\gamma_1, \tilde{\gamma}_2, \dots, \gamma_\rho$. It can be shown that $|\gamma_k| < 1$ $(k = 1, 2, \dots, \rho)$. The proof of this fact is based on the following proposition:

Theorem 2. If

$$F(z) = \frac{1}{2\pi i} \int_E \frac{\zeta + z}{\zeta - z} \sqrt{-\frac{S(\zeta)}{T(\zeta)}} \frac{d\zeta}{\zeta P(\zeta)}$$

is the Carathéodory function corresponding to the special weight, and $C_n(z;P)$ is the polynomial of the second kind, i.e.,

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$$C_n(z;P) = \frac{1}{2\pi i}\int_E \frac{\zeta+z}{\zeta-z}\,[A_n(\zeta;P) - A_n(z;P)]\sqrt{-\frac{S(\zeta)}{T(\zeta)}}\,\frac{d\zeta}{\zeta P(\zeta)},$$

then for $n \geq 2\rho$

$$\frac{C_n(z;P)}{A_n(z;P)} + F(z) = -\frac{i}{P(z)}\left\{\frac{B_n(z;P)}{A_n(z;P)}e^{i\sigma} - \sqrt{-\frac{S(z)}{T(z)}}\right\}.$$

3. The effective construction of the function $p\left(z;\sqrt{R(z)}\right)$ requires the introduction of Abelian integrals. Put

$$h(z,c) = \exp\left\{\frac{1}{2}\int_{e^{i\delta}}^{z}\left[\frac{\sqrt{R(z)}+\sqrt{R(c)}}{z-c} + N_c(z)\right]\frac{dz}{\sqrt{R(z)}}\right\},$$

where $N_c(z)$ is a polynomial of degree ρ with leading coefficient $\sqrt{R(0)}$, whose remaining coefficients are determined by the requirement that the modulus of the function $h(z,c)$ be single-valued on $\mathfrak{F}$. Note that $\ln|h(z,0')|$ is the Green function with respect to the point $z=\infty$ of the domain represented by the first sheet of the surface $\mathfrak{F}$, and

$$\tau = \lim_{z\to\infty}\left|\frac{z}{h(z,0')}\right| = \left|\exp\left\{\frac{1}{2}\int_{e^{i\delta}}^{\infty}\frac{1}{z}\left[1+\frac{N(z)}{\sqrt{R(z)}}\right]dz\right\}\right|,$$

where $N(z) = -\sqrt{R(0)}\,z^{\rho+1} + \cdots + \sqrt{R(0)}$, is the transfinite diameter of the set E. With the aid of the functions $h(z,c)$ one can construct $p\left(z,\sqrt{R(z)}\right)$

$$p\left(z,\sqrt{R(z)}\right) = M\,\frac{[h(z',0')]^n\prod_{k=1}^{2p}h(z,a_k)\prod_{j=1}^{\rho}h(z,\gamma_j)}{\prod_{j=1}^{\rho}h(z,e^{i\alpha_j})[h(z,0)]^p}.$$

To determine the constant M and the normalizing coefficient, introduce the expressions

$$\Gamma_n^*[P] = \left.\frac{\prod_{j=0}^{\rho}|h(z,e^{i\alpha_j})|}{\prod_{j=1}^{\rho}|h(z,\gamma_j)|}\right|_{z=\infty}, \qquad \Gamma_n[P] = \Gamma_n^*[P]\left.\frac{\prod_{j=1}^{\rho}|h(z,\gamma_j)|}{\prod_{j=0}^{\rho}|h(z,e^{i\alpha_j})|}\right|_{z=0},$$

and immediately note that, thanks to the inequalities $|\gamma_k| < 1$ $(k=1,2,\dots,\rho)$, there exists a constant $L>0$ for which

$$\frac{1}{L} < \Gamma_n^*[P] < L.$$

For what follows we still need the function $\Phi(z, f)$, defined by the following conditions: it is regular and different from zero in the domain G, obtains-

···from the plane of z by means of cuts along the arcs E, has a single-valued modulus and its limiting values on E are equal to $f(e^{i\theta})$.

Using condition (1), we find that

$$|M| = 2\tau^n \frac{\Gamma_n^*[P]}{|\Phi(\infty, P)|}, \qquad N_n[P] \equiv \frac{1}{\pi i} \int_E |A_n(z; P)|^2 \sqrt{-\frac{S(z)}{T(z)}} \frac{dz}{zP(z)} =$$

$$= 2\tau^{2n} \frac{\Gamma_n[P]}{P(e^{i\theta})|\Phi(\infty, P)|}.$$

After this, for any $n > 2p$ one can obtain the inequality

$$\left|\sqrt{S(z)}\, A_n(z; P)\Phi(\infty, P)\right| < C\tau^n \sqrt{P(z)} \qquad (z \in E).$$

Here and below, by the letter C we denote various constants depending only on the set E. Further, one can prove that for $z \in E$, $\zeta \in E$, the inequalities

$$\left|\frac{\sqrt{S(z)}\sqrt{S(\zeta)}\, K_n(z, \zeta)}{\sqrt{P(z)}\sqrt{P(\zeta)}}\right| < C, \qquad \left|\frac{K_n(z, \zeta)\sqrt{S(z)}\sqrt{S(\zeta)}}{(z - \zeta)\sqrt{P(z)}\sqrt{P(\zeta)}}\right| < Cn, \tag{2}$$

$$\left|\frac{K_n(z, \zeta)\sqrt{S(z)}}{(z - \zeta)\sqrt{P(\zeta)}\sqrt{P(z)}}\right| < Cn^2,$$

where $K_n(z, \zeta)$ is the usual notation for the Christoffel-Darboux kernel:

$$\pi N_n[P] K_n(z, \zeta) = A_n^*(z)\overline{A_n^*(\zeta)} - A_n(z)\overline{A_n(\zeta)}.$$

4. Theorem 3. If a function $t(z)$, positive on E, has a continuous first derivative, then for any $z \in E$

$$\left|\frac{\sqrt{S(z)}\, A_n(z, t)\Phi(\infty, t)}{\tau^n \sqrt{t(z)}}\right| < C.$$

Theorem 4. If $t(z)$ is a function positive on E and having a continuous first derivative, and $P(z)$ is a quasipolynomial such that

$$\left|1 - \frac{P(z)}{t(z)}\right| < \varepsilon_n \qquad (z \in E),$$

then everywhere on E

$$|A_n(z;t) - A_n(z;P)| \le C\varepsilon_n \sqrt{P(z)}\tau^n |\Phi(\infty, P)|^{-1} n^2.$$

Both theorems follow from inequalities (2) and the equality

$$A_n(z;t) = A_n(z;P) + \int_E A_n(\zeta,t) \frac{K_n(z,\zeta)\sqrt{S(\zeta)}}{\zeta - z} \frac{\sqrt{S(\zeta)}}{\sqrt{R(\zeta)}} \left[\frac{1}{P(\zeta)} - \frac{1}{t(\zeta)}\right] d\zeta.$$

5. Definition. Functions $f_1(z)$ and $f_2(z)$, both positive on E, **belong to one class** if

$$\int_E \frac{S_1(z)}{\sqrt{R(z)}} \frac{\ln f_1(z)}{z - e^{i\alpha_j}}\, dz = \int_E \frac{S_1(z)}{\sqrt{R(z)}} \frac{\ln f_2(z)}{z - e^{i\alpha_j}}\, dz \qquad (j = 1, 2, \dots, p).$$

After this, in the same way as in (4), we obtain the following proposition.

Theorem 5. If

$$f_1(z) = \sum_{k=-p}^{p} A_k z^k, \qquad f_2(z) = \sum_{k=-q}^{q} B_k z^k$$

($f_1(e^{i\theta})$ and $f_2(e^{i\theta}) > 0$ for $e^{i\theta} \in E$) belong to one class, then for $n \ge 2q \ge 2p$ and $z \in G$

$$\frac{A_n(z, f_2) + \dfrac{\sqrt{R(z)}}{S(z)} e^{i\beta} B_n(z; f_2)}{A_n(z; f_1) + \dfrac{\sqrt{R(z)}}{S(z)} e^{i\delta} B_n(z; f_1)} = e^{i\sigma}\Phi\left(z, \sqrt{\frac{f_2}{f_1}}\right) G\left[\sqrt{\frac{f_1}{f_2}}\right],$$

$$\frac{A_n(z; f_2) - \dfrac{\sqrt{R(z)}}{S(z)}\, e^{i\sigma} B_n(z; f_2)}{A_n(z; f_1) - \dfrac{\sqrt{R(z)}}{S(z)}\, e^{i\sigma} B_n(z; f_1)} = e^{i\sigma}\Phi\left(z, \sqrt{\frac{f_1}{f_2}}\right) G\left[\sqrt{\frac{f_1}{f_2}}\right] \frac{f_2(z)}{f_1(z)},$$

where

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$$\Phi(z,\varphi) = \exp \operatorname{Re}\left\{ \frac{1}{\pi i} \int_E \frac{\sqrt{R(z)}}{\sqrt{R(u)}} \frac{\ln \varphi(u)}{z-u}\, du \right\},$$

$$G[\varphi] = \exp \operatorname{Re}\left\{ \frac{\sqrt{R(0)}}{\pi i S_1(0)} \int_E \frac{S_1(u)}{\sqrt{R(u)}} \ln \varphi(u)\, du \right\}.$$

Lemma. Let $t(z)$ $(z \in E)$ be a positive function having a derivative of order r with modulus of continuity $\omega_r(\delta)$. In this case one can construct a quasipolynomial $P(z)$ of degree $\leq n$, belonging to the same class as $t(z)$, and such that

$$\max |\, t(z) - P(z)\, | \leq \frac{A_r}{n^r}\, \omega_r\left(\frac{1}{n}\right),$$

where A_r depends on r, E, and $\max t(z)$ $(z \in E)$.

Theorem 6. Let $t(z)$ $(z \in E)$ have a continuous second derivative, and let $P(z)$ be a quasipolynomial of the same class as $t(z)$. Then, as $n \to \infty$, the following asymptotic equality holds uniformly on E:

$$\frac{A_n(z,t)\Phi(\infty,t)}{\sqrt{t(z)}\,\tau^n G\left[\sqrt{\dfrac{P(z)}{t(z)}}\right]} \sim$$

$$\sim \frac{\Phi(\infty,P)}{\sqrt{P(z)}\,\tau^n} \left\{ A_n(z;P) \cos \Psi_P(z) - e^{i\sigma} \sqrt{-\frac{T(z)}{S(z)}}\, B_n(z,P) \sin \Psi_P(z) \right\},$$

where

$$\Psi_P(z) = \arg \Phi\left(z, \sqrt{\frac{t(z)}{P(z)}}\right).$$

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Note: Figure translations are in progress. See original paper for figures.

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