O. I. ZAVYALOV

1963

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

O. I. ZAVYALOV

MAJORIZATION TECHNIQUE IN QUANTUM MECHANICS

(Presented by Academician N. N. Bogolyubov, 9 II 1963)

Recently there has been growing interest in clarifying the analytic properties of scattering amplitudes in quantum mechanics. In the present note we formulate a majorization method that makes it possible to reduce the investigation of higher terms of the Born expansion to lower orders of perturbation theory. This method is in many respects analogous to the device used in quantum field theory ([1–3]), and in a number of cases leads to a simple proof of one-dimensional dispersion relations in quantum mechanics.

The term of N-th order in the expansion of the amplitude in powers of the smallness of the interaction has the form:

$$f^{(N)}(p^2, \cos\theta) \sim \int e^{-i\mathbf{p}\mathbf{r}_1} V(\mathbf{r}_1) G(\mathbf{r}_1\mathbf{r}_2) V(\mathbf{r}_2) \dots G(\mathbf{r}_{N-1}\mathbf{r}_N) V(\mathbf{r}_N) e^{i\mathbf{p}'\mathbf{r}_N}\, d\mathbf{r}_1 \dots d\mathbf{r}_N.$$

Here $\mathbf{p}$ and $\mathbf{p}'$ are, respectively, the initial and final momenta of the particles, V is the potential describing the interaction, and G is the Green function of the free Schrödinger equation. Passing to momentum space and restricting ourselves to the case of a Yukawa potential with characteristic mass μ (or a superposition of Yukawa potentials), we obtain

Fig. 1

$$f^{(N)}(p^2, \cos\theta) \sim \int d\mathbf{k}_1 \dots d\mathbf{k}_{N-1} \frac{1}{(\mathbf{p}-\mathbf{k}_1)^2+\mu^2} \frac{1}{k_1^2-p^2} \frac{1}{(\mathbf{k}_1-\mathbf{k}_2)^2+\mu^2} \cdots$$

$$\cdots \frac{1}{k_{N-1}^2-p^2} \frac{1}{(\mathbf{k}_{N-1}-\mathbf{p}')^2+\mu^2}. \tag{1}$$

It is convenient to work with formulas of type (1) in terms of diagrams. Thus, the diagram representing the term (1) is shown in Fig. 1. In it, to each solid line there corresponds the Fourier transform of the Green function (i.e., the factor $1/(k^2 - p^2)$); to each dashed line, the Fourier transform of the potential (i.e., the factor $1/(k^2 + \mu^2)$). To each cell of the diagram there corresponds its own "internal" momentum k_i. Integration is carried out over all internal momenta. It is assumed that at the vertices of the diagram not marked by a cross the law of conservation of momentum is satisfied. It is not difficult to verify that the correspondence rules so defined indeed lead to expression (1). Let us use the α-representation. Assign to each horizontal line of the diagram its own parameter α_i (i is the number of this line), and to each potential line the parameter $\eta_{i,i+1}$. Then we write

$$f^{(N)} \sim \int d\mathbf{k}_1 \dots d\mathbf{k}_{N-1} \int_0^\infty \prod d\alpha \prod d\eta \, \exp i \left\{ \sum A_{ij} k_i k_j + 2 \sum B_j k_j p \right\} \times$$

$$\times \exp i \left\{ (\eta_{01} + \eta_{N-1,N}) p^2 + \mu^2 \sum \eta - p^2 \sum \alpha \right\}. \tag{2}$$

Here, as is not hard to see, the coefficients B are respectively equal to:

$B_1 = -\eta_{01}, \; B_2 = 0, \dots, \; B_{N-2} = 0, \; B_{N-1} = -\eta_{N-1,N}$, and the matrix A has the form:

$$\begin{bmatrix} \eta_{01} + \alpha_1 + \eta_{12} & -\eta_{12} & 0 \\ -\eta_{12} & \dots\dots\dots & -\eta_{N-2,N-1} \\ 0 & -\eta_{N-2,N-1} & \eta_{N-2,N-1} + \alpha_{N-1} + \eta_{N-1,N} \end{bmatrix}.$$

Now carrying out the integration over the internal momenta k_i, we arrive at the following expression for the matrix element of the reaction represented by an arbitrary diagram:

$$f^{(N)} \sim \int_0^\infty \prod d\alpha \prod d\eta \, \frac{\varphi(a, \eta)}{Q^c(p^2, \cos\theta, \alpha, \eta)}, \tag{3}$$

where φ is the function of the parameters being integrated, and the denominator Q is given by the equality (for simplicity, the case of forward scattering $p = p'$ is considered, although the arguments below are applicable also to scattering at arbitrary angles):

$$Q(p^2, \alpha, \eta) = \left\{ \eta_{01} + \eta_{N-1,N} - \sum \alpha - \frac{1}{\det A} \left[\eta_{01}^2 \det_{11} + 2\eta_{01}\eta_{N-1,N}(-1)^N \det_{1,N-1} + \eta_{N-1,N}^2 \det_{N-1,N-1} \right] \right\} p^2 + \mu^2 \sum \tag{4}$$

Here $\det_{ij}$ is the determinant of the matrix obtained from A by deleting the i-th column and the j-th row. It is now clear that if, in some region of variation of

p^2, the form Q does not vanish for any positive α and η, then this region is a region of analyticity of $f(p^2)$. Thus, it is easy to verify that for the second-order diagram, at $p^2 < 0$, the denominator Q_2 is strictly positive for all α and η. This corresponds to the usual quantum-mechanical picture of the analytic properties of the amplitude, which assumes only the existence of a cut along the positive half-axis in the p^2 plane.

We shall now prove the following assertion. If for some $p^2 < 0$ the denominator $Q'(p^2, \alpha', \eta')$, corresponding to a diagram of order N, is positive for all α and η, then the denominator $Q(p^2, \alpha, \eta)$ of a diagram of order $(N+1)$ is also strictly positive in this region. This will mean that the second-order diagram is, in a known sense, a majorizing one with respect to all other diagrams. Namely: the higher orders of perturbation theory will certainly be analytic at those points p^2 where the form Q^2 of the lower diagram does not vanish. To prove the formulated proposition it is sufficient to prove that

$$Q(p^2, \alpha, \eta) \geq \min_{\alpha' \eta'} Q'(p^2, \alpha', \eta'). \tag{5}$$

Let us write

$$Q(p^2, \alpha, \eta) \geq p^2 \left\{ \eta_{01} + \eta_{N,N+1} - \sum \alpha \right\} + I(p^2, \alpha, \eta) \Big|_{\eta_{12}=\infty} + \sum_{i \neq 2} \eta_{i-1,i} \mu^2, \tag{6}$$

where

$$I = -\frac{p^2}{\det A} \left[\eta_{01}^2 \det_{11} + 2\eta_{01}\eta_{N,N+1} \prod \eta + \eta_{N,N+1}^2 \det_{NN} \right]. \tag{7}$$

Estimate (6) follows automatically from the easily established fact that $\partial I / \partial \eta_{12} \leq 0$ $(p^2 < 0)$. Passing in (7) to the limit $\eta_{12} \to \infty$, we obtain, from comparing the forms I and I' of the more complicated and the simpler diagrams,

$$I(p^2, \alpha, \eta) \big|_{\eta_{12}=\infty} = I' \left[p^2, \alpha'(\alpha), \eta'(\eta) \right], \tag{8}$$

where it is set

$$\alpha_1' = \alpha_1 + \alpha_2, \quad \eta_{01}' = \eta_{01}, \quad \eta_{12}' = \eta_{23}, \dots, \quad \eta_{i-1,i}' = \eta_{i,i+1},$$

$$\alpha_2' = \alpha_3, \dots, \quad \alpha_{i-1}' = \alpha_i. \tag{9}$$

Fig. 2

Figure 2: Fig. 2

Thus, we have arrived at the inequality

$$Q(p^2, \alpha, \eta) \geq Q'[p^2, \alpha'(\alpha), \eta'(\eta)]. \tag{10}$$

Here the functional dependence $\alpha'(\alpha)$ and $\eta'(\eta)$ is determined by formulas (9). From (10) follows (5), as was required to be proved.

Let us note that the basic estimate (5) can also be obtained without carrying out the calculations (6)–(10). Indeed, increasing the order of the diagram by one is equivalent to replacing the contribution of the first potential and the first solid line by the contribution of an entire cell, multiplied by a factor representing the Green function (Fig. 2). This replacement has the form

$$\exp i\{\eta_{01}[(p-k_1)^2+\mu^2]+\alpha_1'(k_1^2-p^2)\} \to \exp i\left\{p^2\left[\eta_{01}-\frac{\eta_{01}^2}{\eta_{01}+\alpha_1+\eta_{12}}\right]+\right.$$

$$+k_1^2\left[\eta_{12}-\frac{\eta_{12}^2}{\eta_{01}+\alpha_1+\eta_{12}}\right]-2k_1p\frac{\eta_{01}\eta_{12}}{\eta_{01}+\alpha_1+\eta_{12}}+\mu^2(\eta_{01}+\eta_{12})-$$

$$\left.-p^2(\alpha_1+\alpha_2)+\alpha_2k_1^2\right\}. \tag{11}$$

Let us carry out the limiting transition $\eta_{12} \to \infty$. In this case the right-hand side of (11) goes over into

$$\exp i\{\eta_{01}[(p-k_1)^2+\mu^2]+(\alpha_1+\alpha_2)(k_1^2-p^2)+\eta_{12}\mu^2\}.$$

The inequality (5) is now established by a simple comparison of the left- and right-hand sides of (11). At the same time, by increasing η_{12}, we have, as is easy to see, increased the direct quadratic form A. Consequently, the inverse quadratic form A^{-1}, which determines the essential part of the denominator Q, decreased. Therefore the denominator also decreased. Thus it is clear that for finite η_{12} as well the inequality (5) remains valid.

Fig. 2

Let us note that the majorization method described gives nothing new in the case of a simple Yukawa interaction, when the dispersion relations for the amplitude are not difficult to prove by other methods [4]. However, in more complicated problems, for example in the problem of multichannel scattering with a potential of Yukawa type, the technique of majorants proves useful. This problem was posed in work ([5]).

Let us consider it in more detail. Suppose that the particles a_1 and b_1, when colliding with one another, can undergo the following transitions: $a_1 b_1 \to a_1 b_1$; $a_1 b_1 \to a_2 b_2$. Altogether, therefore, there exist 2 reaction modes and 2 independent states of the system. We shall call these independent states channels. Denoting the state vector of the system by $\psi(t)$, we can give it the form of a column with 2 elements. The free Hamiltonian of the system will then be a second-order matrix, and the role of the potential will likewise be played by a second-order matrix. The Schrödinger equation governing the evolution of the system is now written as

$$H_j^0 \psi_j + \sum_{k=1}^{2} V_{jk} \psi_k = i\hbar \frac{\partial}{\partial t} \psi_j.$$

Here

$$H_j^0 = \hat{p}_{aj}^2 / 2m_{aj} + \hat{p}_{bj}^2 / 2m_{bj}.$$

Let V_{ij} have the form of Yukawa potentials:

$$V_{ij} \sim \frac{e^{-\mu_{ij} r}}{r}.$$

Writing the contribution from second-order diagrams in the α-representation, it is easy to find that, in the case of equality of the reduced masses in both channels

$m_{a1} m_{b1} / (m_{a1} + m_{b1}) = m_{a2} m_{b2} / (m_{a2} + m_{b2})$, the denominator Q' in the corresponding integrals is positive for $p^2 < 0$ and for all values of the parameters α, η. If, however, the reduced mass in the 2-m channel is smaller than the reduced mass in the first channel, then already in second order in the coupling constant the denominator Q' can vanish also for negative p^2. It is not difficult to ascertain that such a vanishing proves possible if $p^2 < p_0^2 = -\mu_{12}^2/(1-\sqrt{\lambda})^2$, where λ is the ratio of the reduced masses. Thus an "anomalous" left-hand cut arises in the scattering amplitude along the real axis from $-\mu_{12}^2/(1-\sqrt{\lambda})^2$ to $-\infty$, which does not admit a direct "physical" interpretation in terms of intermediate states. It is clear that a diagram of the type of Fig. 3 will also possess a left-hand cut beginning at the point $-\mu_{22}^2/(1-\sqrt{\lambda})^2$. Thus, if $\mu_{22} < \mu_{12}$, then the anomalous cut for this diagram covers a larger part of the real axis than in the case $\mu_{22} \geqslant \mu_{12}$. But a diagram of the type of Fig. 3 enters as a component part into diagrams of higher orders. Therefore, at first glance it may seem that for $\mu_{22} < \mu_{12}$ the higher terms of the Born series have more noticeable deviations from the traditional picture of analyticity than the lower orders.

Fig. 3: schematic diagram with horizontal axis $\vec{p}$, labels $\kappa^2 - \lambda p^2$, $(\vec{k} - \vec{p})^2 + \mu_{22}^2$, and $(\vec{p} - \vec{k}')^2 + \mu_{12}^2$.

Figure 3: Fig. 3: schematic diagram with horizontal axis $\vec{p}$, labels $\kappa^2 - \lambda p^2$, $(\vec{k} - \vec{p})^2 + \mu_{22}^2$, and $(\vec{p} - \vec{k}')^2 + \mu_{12}^2$.

Fig. 3

It is not difficult to convince oneself, however, that this is not so. Indeed, for $p_0^2 < p^2 < 0$ the denominator Q' of the second-order diagram is positive. In proving the possibility of applying the majorization method, we in no way took into account the properties of the "internal" lines of the diagram. In particular, the argument used does not change also for $\mu_{22} < \mu_{12}$. Therefore, from the inequality $Q' > 0$ and from the assertions proved above it follows that the denominator Q of higher orders will be positive everywhere for $p_0^2 < p^2 < 0$. It is therefore clear that the position of the threshold singularity inducing the anomalous left-hand cut does not change as the number of the Born term increases. Thus it has been proved that, for any orders of perturbation theory, the domain of analyticity of the matrix element as a function of the square of the relative momentum p^2 is a plane with two cuts: from 0 to $+\infty$ and from p_0^2 to $-\infty$.

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
31 I 1963

REFERENCES

1. K. Simanzik, *Progr. Theor. Phys.*, **20**, No. 5 (1958).
2. A. A. Logunov, I. T. Todorov, N. A. Chernikov, *JETP*, **42**, 1285 (1962).
3. A. A. Logunov, I. T. Todorov, N. A. Chernikov, Preprint of the Joint Institute for Nuclear Research, R-889, 1962.
4. Khuri, Treiman, *Phys. Rev.*, **109**, 198 (1958).
5. Fonda, Radicatti, Regge, *Ann. Phys.*, **12**, No. 1 (1961).

Note: Figure translations are in progress. See original paper for figures.