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Abstract

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MATHEMATICS

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THE IMPOSSIBILITY IN THREE-DIMENSIONAL EUCLIDEAN SPACE OF A COMPLETE REGULAR SURFACE WITH A NEGATIVE UPPER BOUND OF THE GAUSSIAN CURVATURE

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§ 1. As is well known, Hilbert proved a theorem according to which in three-dimensional Euclidean space—hereafter denoted by E_3—there does not exist a complete regular surface with negative and constant Gaussian curvature. This means that, upon sufficiently continuing a regular surface of constant negative curvature, we must necessarily encounter a singularity of the surface (a cusp, an edge of regression, etc.).

At the same time, the supposition arose long ago (see, for example, ([1])) that in Hilbert' s theorem the condition of constancy of the curvature is inessential and that the inevitability of the appearance of singularities is caused simply by the circumstance that the Gaussian curvature of the surface is negative and bounded away from zero.

Here it is established that the indicated supposition is correct; namely, that in E_3 a complete regular surface with Gaussian curvature $K \leq \text{const} < 0$ is impossible. This assertion may also be stated in the form of the following theorem:

Theorem. *In E_3, on every complete regular surface the upper bound of the Gaussian curvature is not less than zero.*

The proof of this theorem, which is the main subject of the present note, is given in outline in §§ 4–6.

§ 2. Remark. The present note adjoins our note ([2]), where the asymptotics of incomplete surfaces with a negative upper bound of curvature was investigated; there, in fact, the idea of the proof of the theorem of § 1 is contained, but the conclusion of the theorem was established only for surfaces subject to certain conditions of extrinsic-geometric character. The point is that now we have clarified, to the necessary extent, the question formulated by us at the very end of note ([2]), and this has made it possible to realize the plan outlined then (now already greatly simplified).

Notes ([3–5]) also concern the same topic; there a number of estimate relations between certain intrinsic and extrinsic quantities on a surface of negative curvature were found. With their help, in particular, the conclusion of the theorem of § 1 was proved for surfaces subject to a certain intrinsic-geometric restriction.

§ 3. Explanation of the conditions of the theorem of § 1. As a regularity requirement it is sufficient to assume twice continuous differentiability at every point of the surface. Probably this condition can be somewhat weakened; however, the local homeomorphy of the mapping of the surface onto the Gaussian sphere (by parallelism of normals) must necessarily be ensured. In this connection see ([6]).

Self-intersections and self-overlap of the surface are not prohibited. Completeness of the surface means that, in the sense of its intrinsic geometry, the surface is···

surface is a complete metric space (i.e., every fundamental sequence on it converges).

Item 4. In the proof of the theorem of Item 1, one auxiliary proposition on mappings of two-dimensional domains on the ordinary Euclidean plane is used essentially. It is formulated below in the form of Lemma A; first we shall give a description of the objects of this lemma.

Let there be given in the plane Oxy a simply connected domain D, bounded by a piecewise smooth closed contour Γ. Let Γ contain an arc Γ_1 of positive curvature, whose convexity is directed inward into the domain D; denote the complementary arc by Γ_2. Adjoin to the domain D all points of its boundary Γ, except for a certain point N; we assume the point N to lie on Γ_1, anywhere, but not at the ends of this arc. After such an enlargement of the domain, we shall retain for it the designation by the letter D. Next, let D_0 be an open set, $D \subset D_0$, N does not belong to D_0 (it lies on the boundary), $P^* = \varphi(P)$, $P \in D_0$, a locally homeomorphic mapping of D_0 onto an (open) set D_0^*. We shall agree to regard images as distinct if their preimages are distinct. Thus we consider D_0^* as a many-sheeted domain. Under this understanding of D_0^*, the given mapping is a homeomorphism between D_0 and D_0^*, and, consequently, has an inverse mapping: $P = \psi(P^*)$, $P^* \in D_0^*$. We shall denote the coordinates of the point $P^* \in D_0^*$ by (x, y), and the coordinates of $P = \psi(P^*)$ by (p, q). Then locally the mapping $P = \psi(P^*)$ can be represented by the functions

$$p = p(x, y), \qquad q = q(x, y). \tag{1}$$

We require continuous differentiability of the functions p and q; moreover, we shall assume that the functional determinant of these functions is negative everywhere in D_0^*. Introduce the notation:

$$\Delta = D\begin{pmatrix} p, q \\ x, y \end{pmatrix} = -\varkappa^2, \qquad J = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}.$$

Let us impose on the mapping (1) one more quantitative restriction: suppose there exists a positive constant a ($a = \text{const} > 0$) such that everywhere in D_0^* the inequality

$$\varkappa^2 - a|J| \geq a^2 \tag{2}$$

is satisfied.

Note that $\varkappa$ and J are invariant with respect to rotations of the system of coordinate axes; consequently, condition (2) is also invariant.

Consider $D^* = \varphi(D)$; evidently, $D^* \subset D_0^*$ and it is a many-sheeted domain, to which a part of the boundary has been adjoined (namely, $\varphi(\Gamma \setminus N)$). Introduce in D^* an intrinsic metric, taking as the distance between two points of D^* the exact lower bound of the lengths of rectifiable curves that join these points and pass in D^*. The metric space thus obtained will also be denoted by D^*.

Lemma A. *The metric space D^* is incomplete.*

Remark. For the proof of the theorem of Item 1 only the case $J = 0$ is required.

Remark. If in a neighborhood of the point N the convexity of Γ is directed to the exterior side of the domain D, then even in the case $J = 0$ the assertion of Lemma A may be false.

Item 5. The principal stages of the proof of Lemma A. Without loss of generality one may assume that Γ_1 is an arc of a parabola with vertex at the point N, and Γ_2 is an arc of a circle with center N. Place the origin of coordinates at the point N, direct the axis Ox along the tangent to Γ_1, and the axis Oy along the normal to the exterior side of D.

Let P_0 be the point of intersection of Γ_2 and the axis Oy. We agree on notation: if some object in D is denoted by a certain symbol,

the corresponding object in D^* under the mapping will be denoted by the same symbol, marked with an asterisk.

Suppose that the assertion of the lemma is false. Then the following conclusions can be drawn (some of them do not depend on the assumption made).

1) In D^* there exists a line L^* of infinite length, which starts at the point P_0^* and is a shortest line in D^* between any two of its points; its image in D starts at P_0 and approaches N.

2) Let l be a piecewise smooth arc AB; we shall call its absolute projections onto the Ox-axis and onto the Oy-axis the quantities

$$l_x = \int_{AB} |dx|, \qquad l_y = \int_{AB} |dy|.$$

3) The absolute projection of a shortest line in D^* is no greater than the absolute projection (onto the same axis) of any other line with the same endpoints.

4) Let us call the curves determined by the equation $dp\,dx + dq\,dy = 0$, or

$$2dq = \{-J \pm \sqrt{4\varkappa^2 + J^2}\}\,dx.$$

the **characteristics** of the mapping.

By Peano’ s theorem, through each interior point of D^* there pass at least two characteristics. We shall call the first the one corresponding to the choice of the plus sign before the radical. By condition (2) we have $|dq| \geq a|dx|$.

5) Take an interior point A^* in D^*. At least one of the characteristics has, at the point A^*, a nonvertical direction. We shall move along this characteristic to the left if it is the first, and to the right if it is the second. We shall continue the motion in the same direction as long as this is possible. If at some interior point this possibility is lost, we pass to a characteristic of the other family, and if we had been moving to the left, after that we shall move to the right (and conversely). Thus we construct a certain piecewise smooth line, which we shall call a **chain**, issuing from A^*. When moving in the direction established on the chain, we have $dq \leq 0$; $dq = 0$ only at the ends of the smooth links of the chain.

6) Denote the chain by C^*; we have: $aC^*_x \leq C_y$; $C_y \leq$ const, where the constant depends only on the dimensions of the domain D.

7) $L^*_x \leq +\infty$, $L^*_y = +\infty$.

8) Denote by H the absolute projection onto the Oy-axis of the part of the line L^* from P^*_0 to the current point P^*. Through P^* in D^* draw a horizontal rectilinear segment until its first intersection with the boundary of D^* on the left and on the right (if the point P^* lies on the boundary of D^*, then the segment is drawn only to one side). We shall call the constructed segment a transverse section of D^* if it touches neither L^* nor the boundary of D^*. Denote by $\{H\}$ the set of values of H for which transverse sections are obtained, and the length of a transverse section by $y = y(H)$, $H \in \{H\}$.

9) Take arbitrary $H, H_1 \in \{H\}$, $H_1 < H$, and construct the corresponding transverse sections h^* and h^*_1; let $y = y(H)$, $y_1 = y(H_1)$, $\Delta H = H - H_1$.

In the domain D take a point P on the segment P_0N and draw through it the two tangents to the parabola Γ_1; denote by G the domain enclosed between these tangents and the parabola. Denote by σ the exact lower bound of the area of G under the condition that G contains h. Define σ_1 analogously. Then

$$\Delta H\, \sigma^{1/3} \leq A\sigma_1^{2/3} + B(y + y_1). \tag{3}$$

Here A and B are constants independent of H and H_1. To obtain inequality (3) one must rotate the Ox-axis so that it is directed parallel to the tangent from P to the parabola (P corresponds to σ) and use chains corresponding to both the old and the new axes.

10) For sufficiently large H and H_1, inequality (3) leads to a contradiction.

Item 6. Suppose that in E_3 there exists a complete regular surface F with Gaussian curvature $K \leq \text{const} < 0$. Denote by Ω the universal covering surface of F. Let n be the unit normal to the surface F. Introduce in Ω the local metric $ds^2 = dn^2$; then introduce a metric as a whole by setting the distance between two points of Ω equal to the exact lower bound of the lengths of curves that connect these points in Ω. Denote the resulting metric space by $\widetilde{\Omega}$. It cannot be complete (since it has constant positive Gaussian curvature and is, at the same time, homeomorphic to the Euclidean plane). We complete it as is done in the theory of metric spaces. The set of added elements will be called the boundary of $\widetilde{\Omega}$. By studying the boundary of $\widetilde{\Omega}$ and using Lemma A, one can show that $\widetilde{\Omega}$ is a convex domain on the sphere. From this we obtain a contradiction with Gauss' s theorem on the area of the spherical image, since the area of $\widetilde{\Omega}$ is finite, while the area of Ω with the original metric is infinite. Thus the theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.