



Soviet-era science, translated into English

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1963

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Abstract

Full Text

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OPTIMAL CHOICE OF THE STOPPING TIME OF A MARKOV PROCESS

(Presented by Academician A. N. Kolmogorov on 14 XII 1962)

1. Let a Markov process $(x_t, \xi, \mathcal{M}_t, \mathbf{P}_x)^*$ be given in a phase space E , and let $g(x)$ be a nonnegative function, and let φ_t^s ($0 \leq s < t < \xi$) be a nonnegative additive functional of the process. It is required to choose a random time τ so that the expectation $\mathbf{M}_x\{g(x_\tau) - \varphi_\tau^0\}$ be maximal. Here τ must be a Markov time (m. m.) (i.e., for every t , $\{\tau \leq t\} \in \mathcal{M}_t$). The value $g(x)$ may be interpreted as the gain received if the process is stopped at the point x , and φ_t^s as the cost of observations during the time $[s, t]$. We shall consider in detail the case of a discrete Markov chain, and then indicate the changes that must be made for processes with continuous time.

2. Let the set E be finite or countable, let x_t ($t = 0, 1, 2, \dots$) be a homogeneous Markov chain with transition function $p(x, y)$ ($x, y \in E$), and let

$$\varphi_t^s = \sum_{u=s}^{t-1} c(x_u, x_{u+1})$$

(this is the general form of a homogeneous additive functional). If $G(x) = \mathbf{M}_x \varphi_\xi^0 < \infty$, then for every Markov time τ

$$\mathbf{M}_x \varphi_\tau^0 = G(x) - \mathbf{M}_x G(x_\tau)^{**}.$$

Hence it follows that

$$\mathbf{M}_x [g(x_\tau) - \varphi_\tau^0] = \mathbf{M}_x \tilde{g}(x_\tau) - G(x),$$

where $\tilde{g} = g + G$, and the problem of finding the maximum of $\mathbf{M}_x [g(x_\tau) - \varphi_\tau^0]$ reduces to the problem of finding the maximum of $\mathbf{M}_x \tilde{g}(x_\tau)$. Thus, the case of principal interest is that in which the cost of observations φ_t^s is equal to zero. We shall henceforth consider only this case.

3. Put $\mathbf{P}_t f(x) = \mathbf{M}_x f(x_t)$; in particular,

$$\mathbf{P}_1 f(x) = \mathbf{M}_x f(x_1) = \sum p(x, y) f(y).$$

A function f with values in the interval $[0, \infty]$ is called an excessive function (e. f.) if $\mathbf{P}_1 f(x) \leq f(x)$ for all $x \in E$. We list a number of properties of e. f.

- a) A constant is an e. f.
- b) The exact lower bound of any set of e. f. is again an e. f.
- c) If f is an e. f. and τ is an m. m., then $\mathbf{M}_x f(x_\tau) \leq f(x)$.
- d) If f is an e. f. and $\tau < \tau_1$ are two m. m., then $\mathbf{M}_x f(x_{\tau_1}) \leq \mathbf{M}_x f(x_\tau)$.
- e) If f is an e. f. and τ is the first hitting time of some set Γ , then $f_1(x) = \mathbf{M}_x f(x_\tau)$ is also an e. f.

Properties a) and b) are obvious. We prove c) and d). First suppose f is finite. For any $0 < a < 1$ and any n ,

$$f = h + a\mathbf{P}_1 h + \dots + a^{n-1}\mathbf{P}_{n-1} h + a^n \mathbf{P}_n f,$$

where $h = f - a\mathbf{P}f \geq 0$. Since $a^n \mathbf{P}_n f \leq a^n f \rightarrow 0$ as $n \rightarrow \infty$, we have

$$f = \sum_0^\infty a_n \mathbf{P}_{nh} = \mathbf{M}_x \sum_0^\infty a^n h(x_n).$$

Hence it is clear that

$$\mathbf{M}_x a^{\tau_1} f(x_{\tau_1}) \leq \mathbf{M}_x a^\tau f(x_\tau) \leq f(x).$$

Letting $a \uparrow 1$, we arrive at c) and d). If f is any e. f., then, by virtue of a) and b), $f_n = \min(f, n)$ is a finite e. f., and from the validity of c) and d) for f_n , by passage to the limit, we obtain their validity for the function f . To prove e), denote by τ_1 the mo-

* x_t is the trajectory of the process, ξ is the killing time, \mathcal{M}_t is the collection of events observable during the time $[0, t]$, \mathbf{P}_x is the probability distribution corresponding to the initial state x . (Concerning the terminology and notation, see (1, 2).)

** Even if $\mathbf{M}_x \varphi_\xi^0 = \infty$, such a representation can often be obtained by means of some other function $G(x)$.

the first time after time 1 at which Γ is reached, and note that, by virtue of d),

$$\mathbf{P}_1 f_1(x) = \mathbf{M}_x f_1(x_1) = \mathbf{M}_x \mathbf{M}_{x_1} f(x_\tau) = \mathbf{M}_x f(x_{\tau_1}) \leq f_1(x).$$

4. We shall call a function $s(x)$ an **excessive majorant** (e.m.) of a function $g(x)$ if s is an e.f. and if s is less than or equal to any e.f. $f \geq g$. By virtue of 3, b), to obtain such a function it suffices to take the exact lower bound of all e.f. $f \geq g$. Put $Qf = \max(f, \mathbf{P}_1 f)$. It is not hard to verify that $Q^n g \uparrow s$

as $n \rightarrow \infty$, and this makes it possible to compute the e.m. by the method of successive approximations. On the other hand, if the set E is finite, then the e.m. can be computed by methods of linear programming; indeed, $s(x)$ is equal to the minimum of $f(x)$ on the polyhedron

$$\{f : \mathbf{P}_1 f \leq f, f \geq g\}.$$

Theorem 1. *Let s be the e.m. of the function $g \geq 0$, and let τ_ε be the moment of first hitting of the set $\Gamma_\varepsilon = \{x : g(x) \geq s(x) - \varepsilon\}$. Then the exact upper bound of $\mathbf{M}_x g(x_\tau)$ over all m.m. τ is equal to $s(x)$. If the function g is bounded, then for any $\varepsilon > 0$*

$$s(x) - \varepsilon \leq \mathbf{M}_x g(x_{\tau_\varepsilon}) \leq s(x).$$

If the set E is finite, then $s(x) = \mathbf{M}_x g(x_{\tau_0})$.

Theorem 1 shows that, in the case of a finite set E , the optimal strategy consists in continuing observations until $g(x)$ first becomes equal to $s(x)$. In the countable case an optimal strategy, generally speaking, does not exist; however, for any $\varepsilon > 0$ one can obtain a strategy optimal to within ε . For this it is necessary to continue observations until the inequality

$$g(x) \geq s(x) - \varepsilon$$

is first satisfied.

5. Proof of Theorem 1. Suppose that the function g is bounded. Put $s_\varepsilon(x) \geq \mathbf{M}_x s(x_{\tau_\varepsilon})$. We shall show that $s_\varepsilon(x) \geq g(x)$. Let

$$c = \sup[g(x) - s_\varepsilon(x)] \geq 0.$$

Then the e.f. $s_\varepsilon + c$ (see 3, a) and 3, d)) majorizes g . Therefore $s_\varepsilon + c \geq s$. Let $0 < \alpha < \varepsilon$. There exists $a \in E$ such that

$$g(a) - s_\varepsilon(a) > c - \alpha.$$

Obviously,

$$0 \leq s(a) - g(a) \leq s_\varepsilon(a) + c - g(a) < \alpha.$$

Hence $a \in \Gamma_\alpha \subseteq \Gamma_\varepsilon$, and $s_\varepsilon(a) = \mathbf{M}_a s(x_{\tau_\varepsilon}) = s(a)$. Thus,

$$c - \alpha < g(a) - s_\varepsilon(a) \leq s(a) - s_\varepsilon(a) = 0.$$

Letting $\alpha \downarrow 0$, we observe that $c \leq 0$, and hence $g \leq s_\varepsilon$.

Note that $s_\varepsilon = s$. Indeed, according to 3, c), $s_\varepsilon \leq s$, while from the inequality $g \leq s_\varepsilon$ it follows that $s \leq s_\varepsilon$.

Let τ be an m.m. Since $g \leq s$, we have

$$\mathbf{M}_x g(x_{\tau_\varepsilon}) \leq \mathbf{M}_x s(x_{\tau_\varepsilon}) \leq s(x)$$

(see 3, c)), and consequently

$$s(x) - \varepsilon = \mathbf{M}_x s(x_{\tau_\varepsilon}) - \varepsilon \leq \mathbf{M}_x g(x_{\tau_\varepsilon}) \leq s(x).$$

If the set E is finite, then for all sufficiently small $\varepsilon > 0$ the set Γ_ε coincides with Γ_0 , and, consequently,

$$s(x) - \varepsilon \leq \mathbf{M}_x g(x_{\tau_0}) \leq s(x).$$

It follows that

$$\mathbf{M}_x g(x_{\tau_0}) = s(x).$$

Now let g be unbounded. Denote by s_n the e.m. of the bounded function

$$g_n = \min(g, n).$$

By what has been proved, for any n ,

$$\sup_{\tau} \mathbf{M}_x g(x_{\tau}) \geq \sup_{\tau} \mathbf{M}_x g_n(x_{\tau}) \geq s_n(x).$$

By virtue of 3, c) the left-hand side does not exceed $s(x)$. Since $s_n \uparrow s$, it is equal to $s(x)$.

6. In solving concrete problems the following observation is often useful. In order that $x \in \Gamma_0$, it is sufficient (and necessary) that there be found a “barrier” at the point x , i.e., an e.f. majorizing g and coinciding with g at the point x . On the other hand, in order that $x \notin \Gamma_0$, it is sufficient that

$$\mathbf{P}_1 f(x) > f(x).$$

As an example let us consider the following problem on the optimal choice of one of n objects. It is assumed that after becoming acquainted with any two objects it becomes clear which of them is better. The objects are considered in

random order, and each object considered is either rejected (then one can no longer return to it) or accepted; and then the choice terminates at this point. It is required to choose a strategy that would lead with the greatest probability to the choice of the best object. Number all the objects in the order in which we become acquainted with them.

with them. Put $x_0 = 1$ and denote by x_t ($t \geq 1$) the number of the first object better than the object with number x_{t-1} . It is not hard to verify that $x_0, x_1, \dots, x_t, \dots$ is a Markov chain in the phase space $E = \{1, 2, \dots, n\}$ with transition function $p(k, m) = k/m(m-1)$ for $k < m$, $p(k, m) = 0$ for $k \geq m$. The probability that we make the best choice, stopping at the object $x_t = k$, is equal to k/n . Therefore the problem of optimal choice reduces to the problem of finding the maximum of $\mathbf{M}_x g(x_\tau)$ for $g(x) = x/n$. Let m_n denote the largest integer for which $1/m_n + 1/(m_n + 1) + \dots + 1/(n-1) > 1$. It is not hard to verify that, for $m > m_n$, the function $f(x) = \min(m/n, x/n)$ is a "barrier" at the point m . On the other hand, for $m \leq m_n$, $\mathbf{P}_1 g(m) > g(m)$. Therefore $\Gamma_0 = \{m_n + 1, m_n + 2, \dots, n\}$, and the best strategy is to stop at the first of the objects which turns out to be better than the first m_n objects examined. This prescription was proposed earlier by L. Moser and J. R. Pounder (see (3)), but its optimality apparently had not been proved. We note that $m_n/n \rightarrow 1/e$ as $n \rightarrow \infty$.

7. Suppose now that $(x_t, \xi, \mathcal{M}_t, \mathbf{P}_x)$ is a standard process on the semigroup (E, \mathcal{C}) . The arguments of § 2 are applicable to this case; therefore we shall assume that the cost of observations φ_t^* is equal to zero.

An almost Borel C_0 -lower semicontinuous function $f(x)$ with values in $[0, +\infty]$ is called an excessive function if $\mathbf{P}_t f(x) \leq f(x)$ for all t and x . It is known (see (2) or (4)) that excessive functions have properties 3, a) and 3, c)–3, e). Property 3, b) does not always hold; however, the following theorem is valid.

Theorem 2. Put

$$Q_m f = \max(f, \mathbf{P}_{2-m} f), \quad Q_m^\infty f = \lim_{n \rightarrow \infty} Q_m^n f$$

and

$$Sf = \lim_{m \rightarrow \infty} Q_m^\infty \dots Q_2^\infty Q_1^\infty f.$$

If g is an almost Borel C_0 -lower semicontinuous function, then $s = Sg$ is the excessive minorant of the function g .

Proof. The sequence $Q_m f, Q_m^2 f, \dots$ does not decrease and therefore converges to some limit $Q_m^\infty f$. The sequence $Q_m^\infty \dots Q_2^\infty Q_1^\infty f$ also does not decrease. Therefore it has a limit Sf . All the operators considered map into itself the set of all almost Borel functions. By virtue of (2) (see § 3, Ch. 4) they preserve C_0 -lower semicontinuity. Thus the function $s = Sg$ is almost Borel and C_0 -lower semicontinuous. For any m ,

$$\mathbf{P}_{2-m} Q_m^\infty f \leq Q_m^\infty f.$$

Therefore $\mathbf{P}_{2-m} s \leq s$. In view of the C_0 -lower semicontinuity of the function s , it follows from this that $\mathbf{P}_t s \leq s$ for all t . Thus s is excessive. It is not hard to see that if an excessive $f \geq g$, then $f \geq s$. Hence s is the excessive minorant.

8. Theorem 1 and its proof remain valid for any standard process under the assumption that the function g is C_0 -lower semicontinuous and almost Borel. Only the concluding part of the theorem concerning the conditions under which

$$\mathbf{M}_x g(x_{\tau_0}) = s(x)$$

(and, consequently, τ_0 specifies an optimal strategy) needs modification. It is sufficient, for example, to require that: a) the functions $g(x_t)$ and $s(x_t)$ be almost surely continuous in t ; b)

$$\mathbf{P}_x \{\tau_\varepsilon < \tau_0 = \zeta\} \rightarrow 0$$

as $\varepsilon \downarrow 0$; c) $g(x)$ be bounded. Under these conditions the formula

$$\mathbf{M}_x g(x_{\tau_0}) = \mathbf{M}_x s(x_{\tau_0}) = s(x)$$

can be obtained from the formula

$$\mathbf{M}_x s(x_{\tau_\varepsilon}) = s(x)$$

by passage to the limit as $\varepsilon \downarrow 0$.

9. The problem of optimal stopping of a random sequence $\{x_n\}$ was considered by Snell⁽⁵⁾. Instead of an excessive minorant Snell constructs a minimal supermartingale majorizing $\{x_n\}$. In our case this supermartingale is equal to $s(x_n)$. (This does not follow from Snell's results.)

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Received
12 XII 1962

CITED LITERATURE

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⁴ J. K. Hsiao, *Markov Processes and Potentials*, Moscow, 1962.
⁵ J. L. Snell, *Trans. Am. Math. Soc.*, **73**, 293 (1952).

Note: Figure translations are in progress. See original paper for figures.

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