ON THE GENERALIZATION TO THE SPATIAL CASE OF ONE METHOD FOR SOLVING THE BASIC PLANE PROBLEMS OF POTENTIAL THEORY AND THE THEORY OF ELASTICITY

1963

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MATHEMATICAL PHYSICS

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ON THE GENERALIZATION TO THE SPATIAL CASE OF ONE METHOD FOR SOLVING THE BASIC PLANE PROBLEMS OF POTENTIAL THEORY AND THE THEORY OF ELASTICITY

(Presented by Academician A. Yu. Ishlinskii on 12 VII 1963)

In the paper (^{1}) D. I. Sherman proposed a method for solving plane problems of potential theory for doubly connected domains. In the paper (^{2}) this method was extended to biharmonic problems and applied to the solution of the plane problem of the theory of elasticity. In the present note a generalization of the method is given to spatial problems of potential theory and the theory of elasticity for domains bounded by two surfaces.

Let the domain D_1 be bounded by two surfaces S_1 and S_2, which are Lyapunov surfaces. For definiteness we shall assume that the surface S_1 is situated inside the surface S_2. Denote the points of the domain D_1 and of the infinite domain D_2, bounded by the surface S_2, respectively by p_1 and p_2, and the points of the surfaces S_1 and S_2 by q_1 and q_2. The normal derivative of some function $w(p)$ on the surface S_j $(j = 1, 2)$, depending on whether the derivative is evaluated from inside or from outside S_j, will be denoted by $dw^i(q_j)/dn_j$ or $dw^e(q_j)/dn_j$. We agree to regard the normals to the surfaces as directed from inside D_1 outward.

§ 1. Let us turn to the solution of the Dirichlet problem. It is required to determine a function $u_1(p_1)$, harmonic in the domain D_1, continuous up to the boundary, and assuming on the surfaces S_1 and S_2 the prescribed continuous values $f_1(q_1)$ and $f_2(q_2)$:

$$u_1(q_1) = f_1(q_1) \quad \text{on } S_1; \qquad u_1(q_2) = f_2(q_2) \quad \text{on } S_2. \tag{1}$$

First we solve an auxiliary problem: determine in the domain D_2 a harmonic function $u_2(p_2)$ subject to the boundary condition

$$u_2(q_2) = f_2(q_2) \quad \text{on } S_2. \tag{2}$$

Denote by $\lambda_2(q_2)$ the normal derivative on the surface S_2 of the function $u_2(p_2)$, and introduce on the same surface S_2 an auxiliary function $\lambda_1(q_2)$, which is the value of the normal derivative of the sought solution $u_1(p_1)$. Thus,

$$\lambda_1(q_2) = du_1^i(q_2)/dn_2; \qquad \lambda_2(q_2) = du_2^e(q_2)/dn_2. \tag{3}$$

Next, consider the simple-layer potential

$$V(p) = \int_{S_2} \frac{\nu(q_2)}{r(q_2,p)}\, d\sigma_2; \qquad \nu(q_2) = \frac{\lambda_1(q_2) - \lambda_2(q_2)}{4\pi}. \tag{4}$$

Now define in the domain D_1 the function $u(p_1) = u_1(p_1) - V(p_1)$, and in the domain D_2 the function $u_2'(p_2) = u_2(p_2) - V(p_2)$. From the construction of the function $u_2(p_2)$, in consequence of the continuity of the potential (4) on the surface S_2, it follows that on it the limiting values of the functions $u(p_1)$ and $u_2'(p_2)$ coincide. It is easy to show that on S_2 the limiting values of the normal derivatives of these functions also coincide. Then, on the basis of Stahl' s theorem ([3]), one may assert that the function $u(p_1)$ can be continued into the domain D_2 and there coincides with the function $u_2'(p_2)$, and conversely. Consequently, the function $u(p)$ is harmonic in the domain $D = D_1 + D_2$. On the surface S_1 this

the function assumes the value $f(q_1)$, equal to

$$f(q_1) = f_1(q_1) - \int_{S_2} \frac{\nu(q_2)}{r(q_2,q_1)}\, d\sigma_2, \tag{5}$$

Solving the Dirichlet problem for the domain D with boundary condition (5) (under the assumption that $\nu(q_2)$ is given provisionally), we obtain for the function $u(p)$ the expression

$$u(p) = -\frac{1}{4\pi} \int_{S_1} f(q_1) \frac{dG(p,q_1)}{dn_1}\, d\sigma_1, \tag{6}$$

where $G(p,q_1)$ is the Green function for the infinite domain D.

After this, the sum of the normal derivatives of the functions $u(p)$ and $V(p)$ on the surface S_2 (computed from within D_1) is set, in accordance with (3), equal to the function $\lambda_1(q_2)$. Then, carrying out transformations, we shall obtain, for determining the auxiliary function $\lambda_1(q_2)$, the regular integral equation

$$-\frac{1}{4\pi} \frac{d}{dn_2} \int_{S_1} f(q_1) \frac{dG(q_2,q_1)}{dn_1}\, d\sigma_1 = -\frac{dV^e(q_2)}{dn_2} + \lambda_2(q_2). \tag{7}$$

It can also be written in the form

$$\lambda_1(q_2) - 2\int_{S_2} \nu(q_2')\frac{d}{dn_2}\left(\frac{1}{r(q_2', q_2)}\right) d\sigma_2' -$$

$$-\frac{1}{2\pi}\int_{S_2} \nu(q_2')\int_{S_1} \frac{1}{r(q_2', q_2)}\frac{d^2G(q_2, q_1)}{dn_1 dn_2}\, d\sigma_1\, d\sigma_2' +$$

$$+\frac{1}{2\pi}\int_{S_1} f_1(q_1)\frac{d^2G(q_2, q_1)}{dn_1 dn_2}\, d\sigma_1 + \lambda_2(q_2) = 0. \tag{7'}$$

The solvability of this equation follows from the existence of a solution of the problem under consideration. Having determined from equation (7) the unknown $\lambda_1(q_2)$ and substituted its value into expressions (4) and (6), we then find the desired function $u_1(p_1)$. Indeed, it is not difficult to see that the function $u_1(p_1)$ thus obtained will satisfy the boundary conditions (1). The fulfillment of the first condition is obvious. Consider in the domain D_2 the function $u_2'(p_2)+V(p_2)$ (each of the terms is defined by means of $\lambda_1(q_2)$). It can be shown that the limiting value of the normal derivative on the surface S_2 of this sum is equal to $\lambda_2(q_2)$. From the uniqueness of the solution of the exterior Neumann problem it follows that the function $u_2'(p_2)+V(p_2)$ is identically equal to $u_2(p_2)$ in the domain D_2. Since $u_2'(p)+V(p)$ is continuous when passing through the surface S_2, its limiting value from within the domain D_1 is also equal to $f_2(q_2)$.

It is easy to establish directly (without recourse to the existence theorem) that equation (7) has a unique solution. Let $\lambda_1^{(0)}(q_2)$ be a nontrivial solution of the homogeneous equation corresponding to equation (7) (we arrive at it by setting, in the above reasoning, $f_1(q_1) = f_2(q_2) = 0$). Write this equation in the form

$$du^{(0)}(q_2)/dn_2 = -dV^{(0)}(q_2)/dn_2. \tag{8}$$

The index 0 denotes that the functions introduced above are defined by means of the function $\lambda_1^{(0)}(q_2)$. On the left-hand side of equation (8) is the normal derivative of the function $u^{(0)}(p)$, harmonic in the domain D, and on the right-hand side is the harmonic function in the domain D_2, $-V^{(0)}(p_2)$. From the uniqueness of the solution of the exterior Neumann problem it follows that these functions are identically equal. On the other hand, from the construction of the function $u^{(0)}(p)$ it is clear that on S_1 it coincides with the function $-V^{(0)}(p_1)$. On the surface S_2 it is equal to the function $-V^{(0)}(q_2)$. Thus, the functions $u^{(0)}(p)$ and $V^{(0)}(p_1)$, harmonic respectively in D and in the finite domain bounded by S_2, coincide with one another on the surfaces S_1 and S_2 and, consequently, are analytic continuations of one another respectively into the finite domain bounded by the surface S_1, and into the domain D_2. Since the function $u^{(0)}(p)$ vanishes at infinity

to zero, the functions $u^{(0)}(p)$ and $V^{(0)}(p)$ turn out to be equal to zero. Therefore the density of the potential $V^{(0)}(p)$ (the function $\lambda_1^{(0)}(q_2)$) is identically equal to zero. Thus the homogeneous equation (8) has no nontrivial solutions. It follows from this that equation (7) is always solvable.

Remark 1. In the case when the auxiliary function $\lambda_1(q_1)$ is introduced on the inner surface S_1, so that the domain for which the solution $u_2(p_2)$ is constructed is finite, we arrive at a Fredholm equation (corresponding to equation (7)) that possesses an eigenfunction. Indeed, it is easy to observe that this equation will be homogeneous when $f_2(q_2) = 0$ and $f_1(q_1) = \text{const}$; meanwhile, the solution of the Dirichlet problem for the domain D_1 is not zero. It can be shown that the equation mentioned has only one eigenfunction.

Using arguments analogous to those given above, it is not difficult to arrive at the conclusion that for the function $u_1^*(p_1)$ (constructed from some solution of the integral equation) the boundary condition on the inner surface will, generally speaking, be satisfied only up to a certain constant. For a complete solution of the problem it is necessary to represent the solution of the integral equation as the sum of some particular solution of it and a nontrivial solution of the homogeneous equation multiplied by a certain constant. The value of this constant is to be determined from the first boundary condition (1).

Remark 2. The consideration of the problem in the case when the domain D_1 is infinite is carried out in a similar way.

§ 2. Let us proceed to the solution of the Neumann problem. It is required to determine a function $v_1(p_1)$, harmonic in the domain D_1, under the conditions

$$dv_1(q_1)/dn_1 = \varphi_1(q_1) \quad \text{on } S_1, \qquad dv_1(q_2)/dn_2 = \varphi_2(q_2) \quad \text{on } S_2, \tag{9}$$

where $\varphi_1(q_1)$ and $\varphi_2(q_2)$ are functions prescribed on the surfaces S_1 and S_2 and satisfying certain smoothness conditions. Denote by $\mu_1(q_2)$ the value on the surface S_2 of the sought function $v_1(p_1)$, and by $\mu_2(q_2)$ the value on the same surface of a function harmonic in the domain D_2 and satisfying the second condition (9). Consider the double-layer potential

$$W(p) = \int_{S_2} \nu_1(q_2) \frac{d}{dn_2} \left(\frac{1}{r(p, q_2)} \right) d\sigma_2; \qquad \nu_1(q_2) = \frac{\mu_1(q_2) - \mu_2(q_2)}{4\pi}. \tag{10}$$

It can be shown that the function $v(p) = v_1(p) + W(p)$ is harmonic in the entire domain D. Solving the Neumann problem for the function $v(p)$ (regarding the function $\nu_1(q_2)$ as conditionally prescribed), for example by means of the Neumann function $Q(p, q_1)$, and passing to the function $v_1(p_1)$, we obtain, from the condition of equality on the surface S_2 of the auxiliary function $\mu_1(q_2)$, the regular Fredholm integral equation:

$$\mu_1(q_2) + 2 \int_{S_2} \nu_1(q_2') \frac{d}{dn_2} \left(\frac{1}{r(q_2', q_2)} \right) d\sigma_2'$$

$$-\frac{1}{2\pi}\int_{S_2}\nu_1(q_2')\int_{S_1}\frac{d}{dn_2}\left(\frac{1}{r(q_2',q_2)}\right)Q(q_2,q_1)\,d\sigma_1 d\sigma_2'$$

$$-\frac{1}{2\pi}\int_{S_1}\varphi_1(q_1)Q(q_2,q_1)\,d\sigma_1+\mu_2(q_2)=0. \tag{11}$$

It is not difficult to verify that the homogeneous equation corresponding to equation (11) has one nontrivial solution—a constant.

In conclusion, we note that the function $\varphi_2(q_2)$ must be such that the potential (10) possesses normal derivatives on the surface S_2.

Remark 1. In the case when the auxiliary function is introduced on the surface S_1, it may turn out that the auxiliary problem—the Neumann problem for the finite domain bounded by the surface S_1—is unsolvable unless the condition

$$\int_{S_1}\varphi_1(q_1)\,d\sigma_1=0$$

is satisfied. Then on

to the required solution one must add some harmonic function $w(p)$ in the domain D, satisfying the condition

$$\int_{S_1}\frac{dw(q_1)}{dn_1}\,d\sigma_1=-\int_{S_1}\varphi_1(q_1)\,d\sigma_1.$$

Remark 2. The proposed method can be applied in the case when the value of the function is prescribed on one surface, and the value of its normal derivative on the other.

§ 3. Let us now turn to the solution of the first fundamental problem of the theory of elasticity. Suppose it is required to determine in the domain D_1 the vector of elastic displacement $u_1(u_{1x},u_{1y},u_{1z})$, satisfying the equation

$$\mu\Delta u_1(p_1)+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u_1(p_1)=0 \tag{12}$$

(where λ and μ are the Lamé coefficients) and taking on the surfaces S_1 and S_2 the prescribed continuous values $f_1(q_1)$ and $f_2(q_2)$:

$$u_1(q_1)=f_1(q_1)\quad\text{on } S_1;\qquad u_1(q_2)=f_2(q_2)\quad\text{on } S_2. \tag{13}$$

Introduce on the surface S_2 an auxiliary vector $\lambda_1(q_2)$, formed as a result of applying the so-called operator N to the sought displacement $u_1(p_1)$ (^{4}). Next we determine the vector $u_2(p_2)$, which gives the solution of the problem of elasticity theory for the domain D_2 under the second boundary condition (13). We denote the value of the operator N on the displacement $u_2(p_2)$ by $\lambda_2(q_2)$. Then consider the generalized elastic potential of the first kind (see, for example, (^{5})) ($\Gamma^0(p, q_2)$ is the Somigliana tensor):

$$V(\nu, p) = \int_{S_2} \nu(q_2)\Gamma^0(p, q_2)\, d\sigma_2, \qquad \nu(q_2) = \frac{\mu}{4\pi}\,[\lambda_1(q_2) - \lambda_2(q_2)]. \tag{14}$$

Using the obvious generalization of Stahl' s theorem and the Lyapunov-Tauber theorem as applied to the vector of elastic displacements, one can show that the displacement $u(p_1) = u_1(p_1) - V(\nu, p_1)$ will satisfy equation (12) throughout the whole domain $D = D_1 + D_2$. At the same time, on the surface S_1 the condition

$$u(q_1) \equiv f(q_1) = f_1(q_1) - V(\nu, q_1) \tag{15}$$

is fulfilled (for the time being we regard the vector-function $\nu(q_2)$ as conditionally prescribed). Let the displacement $H(f, p)$ be the solution of the problem of elasticity theory for the domain D under boundary condition (15). Arguing as above, we construct a regular Fredholm vector equation of the second kind for the auxiliary vector-function $\lambda_1(q_2)$. It has the form

$$\lambda_1(q_2) - 2\int_{S_2} \nu(q_2')N_{q_2}(q_2, q_2')\, d\sigma_2' - 2N_{iH}(f, q_2) + \lambda_2(q_2) = 0. \tag{16}$$

It is not difficult to see that the considerations set forth in examining the Dirichlet problem concerning the existence and uniqueness of the solution of integral equation (7) remain, in principle, valid also for the first fundamental problem of elasticity theory.

Remark 1. In the case of the second fundamental problem of elasticity theory, the proposed method, if the value of the displacement vector on one of the surfaces is taken as the auxiliary function (and instead of the potential (14) one introduces a double-layer potential of the first kind with a density determined in a known manner), leads to a vector singular integral equation.

The author expresses his deep gratitude to Prof. D. I. Sherman for his attention to the work and valuable advice.

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Received
5 VII 1963

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Note: Figure translations are in progress. See original paper for figures.

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