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V. D. GOLOVIN

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Abstract

Full Text

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1. Let L^2 be a topological vector space over the field of complex numbers whose elements are measurable functions, defined on the real axis, with integrable square of the modulus on every finite interval; the topology in L^2 is defined by the family of seminorms

$$P_\sigma(f) = \left(\int_{-\sigma}^{\sigma} |f(t)|^2 \, dt \right)^{1/2} \qquad (0 < \sigma < \infty). \tag{1}$$

By L^2_Λ we denote the closed vector subspace generated by the sequence* E_Λ in L^2. The elements of the subspace L^2_Λ are called functions periodic in the mean if $L^2_\Lambda \neq L^2$, i.e., if the sequence E_Λ is not total in L^2. Periodic functions with integrable square of the modulus are obtained from this if for Λ one takes the sequence of all integers.

The general theory of functions periodic in the mean is due to L. Schwartz (^{2}). He proved, in particular, that if for some function $f \in L^2$ the family of elements $T_x(f)$ $(-\infty < x < \infty)$, where T_x is the mapping $f(t) \to f(t+x)$, is not total in L^2, then the closed vector subspace L^2_f generated by this family coincides with some subspace L^2_Λ of functions periodic in the mean; thereby the function f is periodic in the mean, and the sequence Λ is called its spectrum.

The aim of the present note is to study functions periodic in the mean with spectrum forming a regular sequence.

2. Definition 1. A function $f \in L^2$ will be called **regular** if, for some $\sigma > 0$ and every $\tau > \sigma$, there exists a constant M, depending on τ, such that

$$P_\tau \left(\sum c_k T_{\xi_k}(f) \right) \leq M P_\sigma \left(\sum c_k T_{\xi_k}(f) \right) \tag{2}$$

for all finite sequences of complex numbers c_k and real numbers ξ_k.

Theorem 1. *In order that a function $f \in L^2$ be regular, it is necessary and sufficient that f be periodic in the mean with a spectrum that is a regular sequence.*

Indeed, if the function $f \in L^2$ is regular, then the family of elements $T_x(f)$ $(-\infty < x < \infty)$ is not total in L^2, since otherwise for any function $g \in L^2$ the inequality $P_\tau(g) \leq MP_\sigma(g)$ would hold, where M does not depend on g. Consequently, f is a function periodic in the mean, and the subspace L^2_f coincides with the subspace L^2_Λ, where Λ is the spectrum of the function f. By virtue of (2), the functions e_{k_j} of the sequence E_Λ satisfy the inequality

$$P_\tau\left(\sum c_{kj} e_{kj}\right) \leq MP_\sigma\left(\sum c_{kj} e_{kj}\right) \tag{3}$$

for any finite sequence of complex numbers c_{kj}, where the numbers σ, τ, M in inequality (3) are the same as in inequality (2). Thus the sequence Λ is regular and $\sigma \geq \hat{\tau}_\Lambda$ ((1), Theorem 2).

Conversely, let f be a function periodic in the mean and let the regular sequence Λ be its spectrum. Then, for $\sigma > \hat{\tau}_\Lambda$ and any $\tau > \sigma$, there exists a constant M such that inequality (3) holds for any finite sequence of complex numbers c_{kj} ((1), Theorem 1); consequently, inequality (2) also holds. The theorem is proved.

Remark. Let f be a function periodic in the mean and let the regular sequence Λ be its spectrum. The exact lower bound of those $\sigma > 0$ for which, for every $\tau > \sigma$, there exists a constant M, depending on σ and τ, such that for any finite sequences of complex numbers c_k and real numbers ξ_k inequality (2) holds, is equal to $\hat{\tau}_\Lambda$.

3. A sequence of points x_k $(k = 1, 2, ...)$ of a topological vector space E over the field of real or complex numbers is called a **basis** if to each point $x \in E$ there corresponds a unique series with general term $a_k x_k$ (a_k are scalars) converging to x. A basis (x_k) is called a **Riesz basis** if any series with general term $a_k x_k$ (a_k are scalars) converges if and only if $(a_k) \in l^2$.

From the Riesz-Fischer theorem and Parseval' s equality it follows immediately that the sequence of functions e^{ikt} $(k = 0, \pm 1, \pm 2, ...)$ is a Riesz basis in the closed vector subspace generated by it of the space L^2, i.e. in the space of periodic functions with square-integrable modulus.

Theorem 2. *In order that the sequence E_Λ be a Riesz basis in L^2_Λ, it is necessary and sufficient that the sequence Λ satisfy the following conditions:*

I. *The numbers λ_k $(k = 1, 2, ...)$ lie in some strip $|\operatorname{Im}\lambda| \leq h$.*
II. *For some $\delta > 0$ and all $k \neq j$ the inequality*

$$|\lambda_k - \lambda_j| > \delta$$

holds.
III. *The numbers α_k $(k = 1, 2, ...)$ are bounded in the aggregate.*

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[*] We adhere to the terminology and notation of note (^{1}).

Indeed, if the sequence Λ satisfies the listed conditions, then, by virtue of Propositions 3 and 4 of Remark (1), the sequence E_Λ is a Riesz basis in $L^2_\Lambda(-\sigma,\sigma)$ for every $\sigma > \hat{\tau}_\Lambda$; consequently, E_Λ is a Riesz basis in L^2_Λ.

Conversely, if the sequence E_Λ is a Riesz basis in L^2_Λ, then, by Banach' s open mapping theorem, the subspace L^2_Λ is isomorphic to the space l^2. Hence the topology in L^2_Λ can be defined by a norm of the form (1) for some sufficiently large σ, and the sequence E_Λ must be a Riesz basis in $L^2_\Lambda(-\sigma,\sigma)$. Similarly to how this was done in Remark (3), it is now easy to show that conditions I-III hold.

Corollary. *In order that the subspace L^2_f, for $f \in L^2$, possess a Riesz basis of the form E_Λ, it is necessary and sufficient that the function f be regular with a spectrum satisfying condition II.*

4. Following Paley and Wiener (4), we shall call a function $f \in L^2$ **pseudoperiodic** if, for some $\sigma > 0$, there exists a constant M such that

$$P_\sigma\left(\sum c_k T_{x+\xi_k}(f)\right) \le M P_\sigma\left(\sum c_k T_{\xi_k}(f)\right) \tag{4}$$

for any real x and any finite sequences of complex numbers c_k and real numbers ξ_k. The exact lower bound of those $2\sigma > 0$ for which, for the given function f, there exists a constant M possessing the properties indicated above is called the pseudoperiod of

functions f. It is obvious that every pseudoperiodic function is regular (and hence also periodic in the mean). Paley and Wiener showed that a function $f \in L^2$ is pseudoperiodic if and only if it belongs to the class S^2 of almost periodic functions of V. V. Stepanov with discrete spectrum (λ_k), satisfying the condition $\inf|\lambda_k - \lambda_j| > 0$ $(k \ne j)$. In connection with this, B. Ya. Levin posed the question: what conditions must a function $f \in L^2$ satisfy in order that it be periodic in the mean with a **simple spectrum** $\Lambda = (\lambda_k)$, having the properties: 1) all numbers λ_k $(k = 1, 2, ...)$ lie in some strip $|\operatorname{Im}\lambda| \le h$; 2) for some $\delta > 0$ and all $k \ne j$, $|\lambda_k - \lambda_j| > \delta$? Such sequences Λ we shall call **perfectly regular**.

Definition 2. A function $f \in L^2$ will be called **perfectly regular** if, for some $\sigma > 0$, there exists a constant M such that for every function $g \in L^2_f$

$$P_\sigma(T_x(g)) \le M \exp(|x| u(g)) P_\sigma(g), \tag{5}$$

where

$$u(g) = \lim_{|x|\to\infty} \frac{1}{|x|} \ln^+ P_\sigma(T_x(g)). \tag{6}$$

It is clear that every pseudoperiodic function is perfectly regular; it is also obvious that every perfectly regular function is regular.

Theorem 3. *In order that a function $f \in L^2$ be perfectly regular, it is necessary and sufficient that f be periodic in the mean with a spectrum which is a perfectly regular sequence.*

Indeed, if the function f is perfectly regular, then f is a function periodic in the mean with spectrum Λ, which is a regular sequence. Let us prove that the spectrum Λ is simple and satisfies the condition $\inf|\lambda_k - \lambda_j| > 0$ $(k \neq j)$. In fact, if some number λ_k had multiplicity greater than one, then for $g(t) = e_{k2}(t) = ite^{i\lambda_k t}$ $(u(g) = |\operatorname{Im}\lambda_k|)$ we would obtain from (5) a manifestly false consequence. Put $e_k(t) = e^{i\lambda_k t}$ $(k = 1, 2, ...)$. For $k \neq j$ and $h = \max(|\operatorname{Im}\lambda_k|, |\operatorname{Im}\lambda_j|)$ the inequality $u(e_k - e_j) \leq h$ holds. Therefore, when $\delta = |\lambda_k - \lambda_j| \to 0$ and $|x| = 1/\delta$, there exist constants $A, B > 0$, independent of δ, such that

$$A\exp(h/\delta) \leq P_\sigma(T_x(e_k - e_j))$$

and $P_\sigma(e_k - e_j) \leq B\sqrt{\delta}$; this contradicts inequality (5). It is also easy to show that $\sigma > \hat{\tau}_\Lambda$.

Conversely, if $f \in L^2$ is periodic in the mean, with simple perfectly regular spectrum Λ, then for $\sigma > \hat{\tau}_\Lambda$ there exist constants $A, B > 0$ such that (Theorem 2)

$$A\exp(-h_n|x|)\left(\sum|c_k|^2\right)^{1/2} \leq P_\sigma\left(\sum c_k T_x(e_k)\right) \leq$$

$$\leq B\exp(h_n|x|)\left(\sum|c_k|^2\right)^{1/2} \tag{7}$$

for every real x and any complex c_k $(k = 1, 2, ...)$, where h_n is the largest of the numbers $|\operatorname{Im}\lambda_k|$ $(k \leq n)$ for those k for which $c_k \neq 0$. From (7), by Theorem 2, inequality (5) follows.

Corollary. *In order that the subspace L^2_f possess a Riesz basis of the form $E_\Lambda = (e^{i\lambda_k t})$, it is necessary and sufficient that the function f be perfectly regular.*

5. Let the sequence Λ be real. Denote by S_Λ the closed vector subspace generated by the sequence $E_\Lambda = (e^{i\lambda_k t})$ in the space S^2 of V. V. Stepanov. Obviously, $S^2_\Lambda \subset L^2_\Lambda$.

Theorem 4. *The subspaces L^2_Λ and S^2_Λ coincide if and only if*

$$\inf|\lambda_k - \lambda_j| > 0 \quad (k \neq j).$$

If the condition of the theorem is satisfied, then there exist constants $A, B > 0$ such that, for any finite sequence of complex numbers c_k, inequality (7) holds with $h_n = 0$. It follows that the subspaces L^2_Λ and S^2_Λ coincide. Conversely, if the subspaces L^2_Λ and S^2_Λ coincide, then, by the open mapping theorem, the topology in L^2_Λ is determined by a single norm of the form (1), and, moreover, there exists a constant M, independent of x, such that for any function $f \in L^2_\Lambda$ and any finite sequence of complex numbers c_k inequality (4) holds. The assertion now follows from Theorem 3.

Remark. From the theorem just proved and Theorem 3 there follows directly the result of Paley and Wiener, according to which the class of pseudoperiodic functions coincides with the union of the subspaces S^2_Λ for which the sequences Λ satisfy the condition

$$\inf |\lambda_k - \lambda_j| > 0 \quad (k \neq j).$$

Kharkov State University
named after A. M. Gorky

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Note: Figure translations are in progress. See original paper for figures.