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Abstract

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MATHEMATICS

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ON THE APPLICABILITY OF THE FOURIER METHOD TO THE SOLUTION OF A MIXED PROBLEM FOR ONE CLASS OF QUASILINEAR HYPERBOLIC EQUATIONS

(Presented by Academician I. N. Vekua on 1 VIII 1962)

In this note various solutions of the mixed problem for a quasilinear hyperbolic equation of the form

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(\lambda, t, x, u, u_t, u_{x_1}, \dots, u_{x_n}) \tag{1}$$

are investigated, under the initial conditions

$$u\Big|_{t=0} = \varphi(x), \qquad \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \tag{2}$$

and the boundary condition

$$u\Big|_S = 0. \tag{3}$$

Here Ω is an arbitrary n-dimensional domain of points $x = (x_1, x_2, \dots, x_n)$; S is the boundary of this domain; $Q_l = \Omega \times [0 \le t \le l]$ is an $(n+1)$-dimensional cylinder; λ is a parameter; L is a linear self-adjoint operator

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a(x)u, \tag{4}$$

whose coefficients satisfy in Ω the conditions:

$$a(x) \geq 0, \qquad a_{ij}(x) = a_{ji}(x);$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \alpha = \mathrm{const} > 0, \tag{5}$$

where the initial functions $\varphi(x)$ and $\psi(x)$ are prescribed in the domain Ω, while the function $f(\lambda, t, x, u_0, \dots, u_{n+1})$ is prescribed in the domain $G_k = Q_l \times [-k < u_0, \dots, u_{n+1} < k]$ for $|\lambda - \lambda_0| < d$.

First, the following countable system of nonlinear integro-differential equations is studied:

$$u_s(t) = c_s(t) +$$

$$+\frac{1}{\lambda_s}\int_0^t \int_\Omega f[\lambda, \tau, x, u(\tau, x), \dots, u_{x_n}(\tau, x)] v_s(x) \sin\lambda_s(t-\tau)\, dx\, d\tau, \tag{6}$$

where

$$c_s(t) = \varphi_s \cos\lambda_s t + \psi_s \sin\lambda_s t, \qquad \varphi_s = \int_\Omega \varphi(x) v_s(x)\, dx, \tag{7}$$

$$\psi_s = \frac{1}{\lambda_s}\int_\Omega \psi(x) v_s(x)\, dx, \qquad u(t, x) = \sum_{s=1}^{\infty} u_s(t) v_s(x),$$

$\{v_s(x)\}$ are the eigenfunctions, and $\{\lambda_s^2\}$ are the eigenvalues of the operator L.

To investigate system (6) with the aid of functions continuously differentiable on $[0, l]$, we construct all possible sequences

$u(t) = \{u_s(t)\}$, satisfying the condition:

$$\sum_{s=1}^{\infty}\left\{\left[\lambda_s^\alpha \max_{0\leq t\leq l}|u_s(t)|\right]^p + \left[\lambda_s^\beta \max_{0\leq t\leq l}|\dot{u}_s(t)|\right]^p\right\} < +\infty,$$

where $\alpha \geq 0,\ \beta \geq 0,\ p \geq 1$. We shall denote this set by $B_p^{(\alpha,\beta)}(0, l)$ and define the norm

$$\|u(t)\| = \left(\sum_{s=1}^{\infty}\left[\lambda_s^\alpha \max_{0\leq t\leq l}|u_s(t)|\right]^p\right)^{1/p} + \left(\sum_{s=1}^{\infty}\left[\lambda_s^\beta \max_{0\leq t\leq l}|\dot{u}_s(t)|\right]^p\right)^{1/p};$$

$B_p^{(\alpha,\beta)}(0, l)$ will be a Banach-type space.

1. Generalized solution. Here two theorems are given on the existence and uniqueness of a generalized solution of problem (1), (2), (3) in the sense of paper (3^a). In addition, the continuous dependence of the solution on the initial functions φ, ψ and the function f is studied.

For the existence and uniqueness of a generalized solution the following holds:

Theorem 1. *Suppose:*

a) Ω is an arbitrary n-dimensional bounded connected domain; the coefficients of the operator L are measurable, bounded on Ω, and satisfy conditions (5);

b) $\varphi(x) \in \overset{0}{D}(\Omega)$ *, $\psi(x) \in L_2(\Omega)$;

c) $f(\lambda, t, x, 0, \dots, 0) \in L_2(Q_l)$, the function f is measurable in (t, x) for all values of $\lambda, u_0, \dots, u_{n+1}$ and satisfies in G_∞ the condition

$$|f(\lambda, t, x, u_0, \dots, u_{n+1}) - f(\lambda, t, x, \tilde{u}_0, \dots, \tilde{u}_{n+1})| \leq \mu(t) \sum_{i=0}^{n+1} |u_i - \tilde{u}_i|, \quad (8)$$

where $\mu(t) \in L_2(0, l)$.

Then problem (1), (2), (3) has a unique generalized solution.

The theorem is proved analogously to Theorems 1 and 2 of paper (3^a), in contrast to which here the existence and uniqueness of the solution of system (6) in $B_2^{(1,0)}(0, l)$ is proved.

The following theorems are more general theorems on the existence of a generalized solution of problem (1), (2), (3).

Theorem 2. *Suppose conditions a), b) of Theorem 1 are satisfied. Then, if the function f satisfies the Carathéodory conditions and in the domain G_∞ the condition*

$$|f(\lambda, t, x, u_0, \dots, u_{n+1})| \leq \mu(t) \sum_{i=0}^{n+1} |u_i| + b(t, x), \quad (9)$$

where $\mu(t) \in L_2(0, l)$, $b(t, x) \in L_2(Q_l)$, then problem (1), (2), (3) has at least one generalized solution.

The following theorem holds on the continuous dependence of the solution of problem (1), (2), (3) on the initial functions and the function f.

Theorem 3. *Let the function $\tilde{u}(t, x)$ be a solution of the mixed problem*

$$\frac{\partial^2 u}{\partial t^2} = Lu + f(\lambda, t, x, u, u_t, u_{x_1}, \dots, u_{x_n}) + \tilde{f}_0(t, x), \quad (10)$$

$$u\big|_{t=0} = \varphi(x) + \tilde{\varphi}(x), \qquad \frac{\partial u}{\partial t}\bigg|_{t=0} = \psi(x) + \tilde{\psi}(x), \qquad u\big|_S = 0.$$

* The definition of all classes used may be found, for example, in (3).

If the conditions of Theorem 1 are satisfied, and the norms of the functions $\widetilde{\varphi}(x)$, $\widetilde{\psi}(x)$, and $\tilde{f}_0(t,x)$, respectively in the sense of $W_2^{(1)}(\Omega)$, $L_2(\Omega)$, $L_2(Q_l)$, differ little from zero, then the solutions of problems (1), (2), (3) and (10) differ little in the sense of $W_2^{(1)}(\Omega,t)$.

The proof of the theorem follows from the inequality

$$\|u(t,x)-\tilde{u}(t,x)\|^2_{W_2^{(1)}(\Omega,t)} \leq$$

$$\leq c\left\{\|\widetilde{\varphi}(x)\|^2_{W_2^{(1)}(\Omega)}+\|\widetilde{\psi}(x)\|^2_{L_2(\Omega)}+\|\tilde{f}_0(t,x)\|^2_{L_2(Q_l)}\right\}\exp\left(c\int_0^t \mu^2(\tau)\,d\tau\right),$$

where c is a certain constant number.

2. Solution almost everywhere. By a solution almost everywhere of problem (1), (2), (3) we shall understand it in the same sense as in [36]. We formulate two theorems on the existence and uniqueness of a solution almost everywhere. The first theorem is local in character, i.e., it holds for sufficiently small $|\lambda-\lambda_0|$ or small l; the second theorem is nonlocal in character.

Theorem 4. Suppose:

1) Ω is an arbitrary normal three-dimensional domain contained, together with its boundary S, in some open domain C; the coefficients of the operator L belong to the classes

$$a_{ij}(x)\in C^{(1,\mu)}, \qquad a(x)\in C^{(0,\mu)} \qquad (\mu>0) \tag{11}$$

and satisfy conditions (5) in C;

2) the initial functions $\varphi(x)\in W_2^{(2)}(\Omega)$, $\varphi(x),\psi(x)\in \mathring{D}(\Omega)$;

3) the function $f(\lambda,t,x,u_0,\dots,u_4)$ has partial derivatives with respect to x_i and with respect to u_i, and in the domain G_k satisfies the conditions:

$$\left|f'_{x_i}(\lambda,t,x,u_0,\dots,u_4)-\tilde{f}'_{x_i}(\lambda,t,x,\tilde{u}_0,\dots,\tilde{u}_4)\right| \leq$$

$$\leq b_i(\lambda,t,x)|u_0-\tilde{u}_0|+b_i(\lambda,t)\sum_{j=1}^4|u_j-\tilde{u}_j|,$$

$$\left|f'_{u_i}(\lambda,t,x,u_0,\dots,u_4)-f'_{u_i}(\lambda,t,x,\tilde{u}_0,\dots,\tilde{u}_4)\right| \leq b(\lambda,t)|u_0-\tilde{u}_0|,$$

where $b(\lambda,t)$, $b_i(\lambda,t)$, $b_i(\lambda,t,x)$ belong to L_2 for $|\lambda-\lambda_0|<d$, are continuous in L_2 with respect to the parameter λ in some neighborhood of λ_0, and tend to zero as $\lambda\to\lambda_0$; $f'_{u_i}(\lambda,t,x,0,\dots,0)\to 0$ as $\lambda\to\lambda_0$,

$$\sup_{x\in\Omega}\left[f'_{u_i}(\lambda,t,x,0,\dots,0)\right]^2\in L(0,l);$$

4) $f(\lambda,t,x,0,0,u_2,u_3,u_4)\in \overset{\circ}{D}_1(Q_l)$.

Then, for sufficiently small values of l for any λ, or for sufficiently small $|\lambda-\lambda_0|$, there exists a unique solution almost everywhere of problem (1), (2), (3).

To prove this theorem it is first necessary to prove the existence of a solution of system (6) in $B_2^{(2,1)}(0,l)$.

Theorem 5. Suppose the domain Ω, the coefficients of the operator L, and the initial functions $\varphi(x)$, $\psi(x)$ satisfy the same requirements as in Theorem 4. If, for each fixed λ,u_2,u_3,u_4, the function

$$f(\lambda,t,x,0,0,u_2,u_3,u_4)$$

belongs to the class $\overset{\circ}{D}_1(Q_l)$, and the function f has partial derivatives with respect to x_i and with respect to u_i, satisfying in the domain G_∞ the conditions

$$|f(\lambda,t,x,u_0,\dots,u_4)|\le b(\lambda,t,x)|u_0|+b(\lambda,t)\sum_{i=1}^{4}|u_i|+\tilde{b}(\lambda,t,x);$$

$$|f'_{x_i}(\lambda,t,x,u_0,\dots,u_4)|\le b_i(\lambda,t,x)|u_0|+b_i(\lambda,t)\sum_{j=1}^{4}|u_j|+\tilde{b}_i(\lambda,t,x),$$

$$(i=1,2,3);$$

$$|f'_{u_i}(\lambda,t,x,u_0,\dots,u_4)|\le c_i(\lambda,t)|u_0|+\tilde{c}_i(\lambda,t)\qquad (i=0,1,2,3,4),$$

where $b(\lambda,t)$, $b_i(\lambda,t)$, $c_i(\lambda,t)$, $\tilde{c}_i(\lambda,t)\in L_2(0,l)$; $b(\lambda,t,x)$, $\tilde{b}(\lambda,t,x)$, $b_i(\lambda,t,x)\in L_2(Q_l)$ for $|\lambda-\lambda_0|<d$, then there exists at least one solution almost everywhere of problem (1), (2), (3).

Remark. If for a generalized solution $u(t,x)$ the function

$$g(\lambda,t,x)=f[\lambda,t,x,u(t,x),\dots,u_{x_n}(t,x)]$$

belongs to the class $\overset{0}{D}_1(Q_l)$, then the generalized solution is at the same time a solution almost everywhere.

3. Classical solution. A classical solution of problem (1), (2), (3) is a function $u(t,x)$ having in $\overline{Q}_l$ continuous derivatives up to the second order and satisfying conditions (1), (2), (3) in the ordinary classical sense.

Suppose that the following conditions are satisfied:

1) Ω is an arbitrary normal three-dimensional domain contained, together with its boundary S, in some open domain C; the coefficients of the operator L belong to classes (11) in the domain C and satisfy conditions (5) in C. Moreover, let the boundary of the domain Ω and the coefficients of the operator L satisfy such smoothness conditions that the series

$$\sum_{s=1}^{\infty}\left[\frac{v_s(x)}{\lambda_s^2}\right]^2, \qquad \sum_{s=1}^{\infty}\left[\frac{\partial v_s(x)/\partial x_i}{\lambda_s^3}\right]^2, \qquad \sum_{s=1}^{\infty}\left[\frac{\partial^2 v_s(x)/\partial x_i \partial x_j}{\lambda_s^4}\right]^2 \tag{12}$$

converge uniformly in the closed domain $\overline{\Omega}$, and the eigenfunctions $v_s(x)$ have continuous derivatives up to the fourth order inside Ω.

2) $\varphi(x) \in W_2^{(4)}(\Omega)$, $\psi(x) \in W_2^{(3)}(\Omega)$, and $\varphi(x)$, $L\varphi(x)$, $\psi(x)$, $L\psi(x) \in \overset{0}{D}(\Omega)$.

3) The function $f(\lambda, t, \xi_1, \dots, \xi_8)$ has partial derivatives with respect to ξ_i up to the third order inclusive and in G_k satisfies the conditions

$$\left|\frac{\partial^3 f(\lambda, t, \xi_1, \dots, \xi_8)}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2} \cdots \partial \xi_8^{\alpha_8}} - \frac{\partial^3 f(\lambda, t, \xi_1, \xi_2, \xi_3, \tilde{\xi}_4, \dots, \tilde{\xi}_8)}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2} \cdots \partial \xi_8^{\alpha_8}}\right| \le$$

$$\le b_{\alpha_1 \alpha_2 \dots \alpha_8}(\lambda, t, \xi_1, \xi_2, \xi_3) \sum_{i=4}^{8} |\xi_i - \tilde{\xi}_i|,$$

where $\alpha_1 + \alpha_2 + \dots + \alpha_8 = 3$; $b_{\alpha_1 \alpha_2 \dots \alpha_8}(\lambda, t, \xi_1, \xi_2, \xi_3) \in L_2(Q_l)$ for $|\lambda - \lambda_0| < d$.

4) For every λ, u_2, u_3, u_4, the functions

$$f(\lambda, t, x, 0, 0, u_2, u_3, u_4),$$

$$Lf(\lambda, t, x, 0, 0, u_2, u_3, u_4)$$

belong to the class $\overset{0}{D}_1(Q_l)$.

Theorem 6. *Let conditions 1, 2, 3, 4 be satisfied. Then, for sufficiently small values of l, there exists a unique classical solution of problem (1), (2), (3).*

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Note: Figure translations are in progress. See original paper for figures.