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**Abstract**

**Full Text**

**MATHEMATICS**

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**EXPANSION IN EIGENFUNCTIONS OF  
NON-SELFADJOINT SINGULAR SECOND-  
ORDER DIFFERENTIAL OPERATORS DE-  
PENDING ON A PARAMETER**

*(Presented by Academician I. N. Vekua on 16 VII 1963)*

M. A. Naimark <sup>(1)</sup> investigated the spectrum and expansions in eigenfunctions of a non-selfadjoint singular differential operator  $L_\theta$ , generated on the half-axis  $R^+ = [0, \infty)$  by the differential expression

$$l(y) \equiv -y'' + p(x)y, \quad 0 \leq x < +\infty, \quad (1)$$

and by the boundary condition

$$y'(0) - \theta y(0) = 0, \quad (2)$$

where  $p(x)$  is a complex-valued function such that  $x^2 p(x) \in L(R^+)$ , and  $\theta$  is a complex number.

In the present paper, by M. A. Naimark's method, the expansion in eigenfunctions of the operator  $L_{\theta(\lambda)}$  is studied in the case when, in the boundary conditions (2),  $\theta$  depends on the complex parameter  $\lambda$ .

The case in which the differential expression  $l(y)$  is selfadjoint in the sense of Lagrange was considered by A. V. Shtraus <sup>(2)</sup>. Let  $p(x)$  be a complex-valued function summable on each finite interval  $[0, a]$ ,  $a > 0$ .

Denote by  $D$  the set of all functions  $y(x) \in L^2(R^+)$  satisfying the conditions: 1)  $y(x)$  and  $y'(x)$  are absolutely continuous on each finite interval  $[0, a]$ ,  $a > 0$ ; 2)  $l(y) \in L^2(R^+)$ . Let  $\theta(\lambda) = P(\lambda)/Q(\lambda)$ , where  $P(\lambda)$  and  $Q(\lambda)$  are polynomials in powers of the complex parameter  $\lambda$ , and moreover

$$\lim_{\lambda \rightarrow s^2} \frac{\theta(\lambda)}{is} \neq 1, \quad \lambda = s^2.$$

Denote by  $D_{\theta(\lambda_0)}$  the set of all functions  $y(x) \in D$  satisfying, at the fixed point  $\lambda_0$ , the condition\*

$$y'(0) - \theta(\lambda_0)y(0) = 0. \quad (3)$$

Denote by  $L_{\theta(\lambda_0)}$  the operator with domain  $D_{\theta(\lambda_0)}$  such that, for  $y \in D_{\theta(\lambda_0)}$ ,  $L_{\theta(\lambda_0)}y = l(y)$ .

If  $\lambda$  runs through the set of all points of the  $\lambda$ -plane, then we obtain a family of non-selfadjoint singular operators  $L_{\theta(\lambda)}$ , depending on the parameter  $\lambda$ . Let  $p(x) \in L(R^+)$ .

**Theorem 1.** *The spectrum of the operator  $L_{\theta(\lambda)}$  is continuous on the positive half-axis and discrete in the entire remaining complex  $\lambda$ -plane. The eigenvalues of the operator  $L_{\theta(\lambda)}$  form a bounded set, whose limit points may lie only on a finite interval of the positive half-axis  $\lambda \geq 0$ . For values of  $\lambda$  not belonging to the spectrum, the resolvent  $R_\lambda = (L_{\theta(\lambda)} - \lambda I)^{-1}$  of the operator  $L_{\theta(\lambda)}$  is an integral operator with kernel  $K(x, t, \lambda)$ , satisfying the conditions*

$$\int_0^\infty |K(x, t, \lambda)|^2 dx < \infty, \quad \int_0^\infty |K(x, t, \lambda)|^2 dt < \infty. \quad (4)$$

\* If  $\lambda_0$  is a pole of  $\theta(\lambda)$ , then condition (3) means  $y(0) = 0$ .

Suppose that for some  $\varepsilon > 0$  the function  $p(x)$  satisfies the condition

$$\int_0^\infty e^{\varepsilon x} |p(x)| dx < \infty. \quad (5)$$

Then, as M. A. Naimark showed <sup>(1)</sup>, the differential equation  $l(y) = \lambda y$  has solutions  $y_1(x, s)$ ,  $y_2(x, s)$ , where  $y_1$  and  $y_2$  are holomorphic functions of  $s$  in the half-planes  $\text{Im } s > -\frac{1}{2}\varepsilon$ ,  $\text{Im } s < \frac{1}{2}\varepsilon$ , respectively. As  $x \rightarrow \infty$ ,

$$y_1(x, s) = e^{ixs} [1 + o(1)], \quad y_1'(x, s) = e^{ixs} [is + o(1)]; \quad (6a)$$

$$y_2(x, s) = e^{-ixs} [1 + o(1)], \quad y_2'(x, s) = e^{-ixs} [-is + o(1)]; \quad (6b)$$

as  $s \rightarrow \infty$  and  $\text{Im } s \rightarrow 0$ , and on the positive half-axis  $s \geq 0$ , respectively,

$$y_1(x, s) = e^{ixs} \left[ 1 + O\left(\frac{1}{s}\right) \right], \quad y_1'(x, s) = ise^{ixs} \left[ 1 + O\left(\frac{1}{s}\right) \right]; \quad (7a)$$

$$y_2(x, s) = e^{-ixs} \left[ 1 + O\left(\frac{1}{s}\right) \right], \quad y_2'(x, s) = -ise^{-ixs} \left[ 1 + O\left(\frac{1}{s}\right) \right]. \quad (7b)$$

Here the asymptotic formulas (6a) and (6b) hold uniformly with respect to  $s$  in the regions  $\operatorname{Im} s > -\varepsilon_1$ ,  $\operatorname{Im} s < \varepsilon_1$ ,  $\varepsilon_1 < \frac{1}{2}\varepsilon$ , while formulas (7a) and (7b) hold for  $\operatorname{Im} s > -\frac{1}{2}\varepsilon$ ,  $\operatorname{Im} s < \frac{1}{2}\varepsilon$ , respectively, uniformly with respect to  $x$  from the interval  $R^+$ .

Assume that the functions

$$A(s) = y_1'(0, s) - \theta(\lambda)y(0, s),$$

$$\tilde{A}(s) = y_2'(0, s) - \theta(\lambda)y(0, s)$$

do not vanish on the half-axis  $s \geq 0$ ; then, from the expression for  $\theta(\lambda)$  and from formulas (7a) and (7b), it follows that  $A(s)$  and  $\tilde{A}(s)$  do not vanish in the half-strip  $|\operatorname{Im} s| < \varepsilon_1$ ,  $\operatorname{Re} s \geq 0$ . Therefore, when condition (5) is fulfilled, the spectrum of the operator  $L_{\theta(\lambda)}$  can have only a finite number of eigenvalues. None of the eigenvalues of the operator  $L_{\theta(\lambda)}$  lies on the half-axis  $\lambda \geq 0$ . Here  $\lambda = 0$  belongs to the continuous spectrum. Suppose that the eigenvalues of the operator  $L_{\theta(\lambda)}$  are simple poles of its resolvent.

Consider the auxiliary boundary-value problem on the interval

$$-y'' + p(x)y = \lambda y, \quad y'(0) - \theta(\lambda)y(0) = 0, \quad y(b) = 0. \quad (8)$$

For sufficiently large  $b$ , to each of the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_r$  of the operator  $L_{\theta(\lambda)}$  there corresponds exactly one eigenvalue  $\lambda_1(b), \lambda_2(b), \dots, \lambda_r(b)$  of the boundary-value problem (7), (8), so that  $\lambda_k(b) \rightarrow \lambda_k$  as  $b \rightarrow \infty$ . All the remaining eigenvalues of the auxiliary boundary-value problem (7), (8) satisfy the relations

$$\lambda = s^2, \quad s = \frac{n\pi}{b} + \frac{1}{2ib}\omega\left(\frac{n\pi}{b}\right) + o\left(\frac{1}{b}\right) \quad \text{as } b \rightarrow \infty$$

uniformly with respect to  $s$  in the half-strip  $|\operatorname{Im} s| < \varepsilon_1$ ,  $\operatorname{Re} s \geq 0$ , where  $\omega(s) = \ln \frac{A(s)}{\tilde{A}(s)}$  is a single-valued holomorphic function. Let  $y_{1,b}(x), y_{2,b}(x), \dots, y_{r,b}(x)$  be the eigenfunctions of the boundary-value problem (7), (8), corresponding to the eigenvalues  $\lambda_1(b), \lambda_2(b), \dots, \lambda_r(b)$ . Then, as  $b \rightarrow \infty$ ,

$$\frac{y_{k,b}(x, s)}{\int_0^\infty [y_{k,b}(x)]^2 dx} = \frac{y_k(x)}{\int_0^\infty [y_k(x)]^2 dx} + o(1)$$

uniformly with respect to  $x \in [0, c]$ ,  $c > 0$ . Moreover, if  $\tilde{\lambda} = |\tilde{s}(b)|^2$  are eigenvalues of the problem (7), (8) such that  $\operatorname{Im} s \rightarrow 0$  as  $b \rightarrow \infty$ , then, as  $b \rightarrow \infty$ ,

$$\frac{1}{b} \int_0^\infty [\tilde{y}(k, s)]^2 dx = -2A(\tilde{s})\tilde{A}(\tilde{s}); \quad \tilde{y}(x, s) = \tilde{A}(s)y_1(x, s) - A(s)y_2(x, s)$$

uniformly with respect to  $\tilde{s}$  in every rectangle

$$|\operatorname{Im} \tilde{s}| < \varepsilon_1 < \frac{1}{2}\varepsilon, \quad \alpha \leq \operatorname{Re} \tilde{s} \leq \beta, \quad 0 \leq \alpha < \beta.$$

Let  $K_b(x, t, \lambda)$  be the resolvent kernel of the auxiliary boundary-value problem (7), (8). If  $\lambda$  is not an eigenvalue of the operator  $L_{\theta(\lambda)}$ , then, as  $b \rightarrow 0$ ,

$$K_b(x, t, \lambda) = K(x, t, \lambda) + o(1) \quad (9)$$

uniformly with respect to  $x, t$  in every finite square  $0 \leq x, t \leq c, c > 0$ . We shall consider the kernel  $K_b(x, t, \lambda)$  on the curves  $C_{m,q}$  ( $m, q$  are natural numbers), defined as follows:  $C_{m,q}$  is a closed curve composed of two arcs  $C'_{m,q}, C''_{m,q}$ , consisting of points  $\lambda$  satisfying the relations

$$C'_{m,q}: \quad \lambda = s^2, \quad |s|^2 = R_{m,q}^2 + i\tau, \quad -\varepsilon_1 \leq \tau \leq \varepsilon_1, \quad R_{m,q} = \left(m + \frac{1}{2q}\right) \frac{\tau}{\sqrt{m}};$$

$$C''_{m,q}: \quad \lambda = s^2, \quad |s|^2 = R_{m,q} + \varepsilon_1^2, \quad \operatorname{Im} s > \varepsilon_1. \quad (10)$$

On the curves  $C_{m,q}$ ,  $|K_b(x, t, \lambda)| \leq C/\sqrt{\lambda}$ , and, consequently,

$$\frac{1}{2\pi i} \oint_{C_{m,q}} \frac{K_b(x, t, \lambda)}{\lambda - \lambda_0} d\lambda \rightarrow 0 \quad \text{as } m \rightarrow \infty \quad (11)$$

uniformly with respect to  $q$ .

Applying the residue theorem to this integral, passing to the limit as  $m \rightarrow \infty$ ,  $q \rightarrow \infty$ , and taking into account the relations (9) and (10), we obtain

$$K(x, t, \lambda) = \sum_{k=1}^r \frac{y_k(x)y_k(t)}{(\lambda_k - \lambda) \int_0^\infty [y_k(x)]^2 dx} - \frac{1}{2\pi} \int_0^\infty \frac{\tilde{y}(x, s)\tilde{y}(t, s)}{(s^2 - \lambda)A(s)\tilde{A}(s)} ds. \quad (12)$$

Approximating an arbitrary function  $p(x)$  satisfying the condition  $x^2 p(x) \in L(R^+)$  by functions  $p(x)$  satisfying condition (4), we arrive at the following result.

**Theorem 2.** Let  $x^2 p(x) \in L(R^+)$ , and let the functions  $A(s)$ ,  $\tilde{A}(s)$  not vanish for  $s \geq 0$ ; let the eigenvalues of the operator  $L_{\theta(\lambda)}$  be simple poles of its resolvent, and let

$$\theta(\lambda) = \frac{P(\lambda)}{Q(\lambda)}, \quad \lim_{s \rightarrow \infty} \frac{\theta(\lambda)}{is} \neq 1.$$

Then, for every point  $\lambda$  not belonging to the spectrum of the operator  $L_{\theta(\lambda)}$ , equality (12) holds, in which the integral on the right-hand side converges absolutely and uniformly with respect to  $x, t$  in the domain  $0 \leq x, t < \infty$ .

Denote by  $M$  the totality of all functions  $g(x) \in L(R^+)$  satisfying the conditions: 1)  $g'(x)$  exists and is absolutely continuous in every finite interval  $[0, a]$ ,  $a > 0$ ; 2)  $l(g) \in L(R^+)$ ; 3)  $g(0) = 0$ ,  $g'(0) = 0$ . Multiplying both sides of (12) by the function  $f(x) = l(g) - \lambda g$  and integrating with respect to  $x$  from 0 to  $\infty$ , we obtain:

**Theorem 3.** Under the conditions of Theorem 2, every function  $g(x) \in M$  can be represented in the form

$$g(x) = \sum_{k=1}^r \frac{\alpha_k y_k(x)}{\int_0^\infty [y_k(t)]^2 dt} - \frac{1}{2\pi} \int_0^\infty \frac{\alpha(s) \tilde{y}(x, s)}{A(s) \tilde{A}(s)} ds, \quad (13)$$

where

$$\alpha_k = \int_0^\infty g(t) y_k(t) dt, \quad \alpha(s) = \int_0^\infty g(t) \tilde{y}(t, s) dt,$$

and the integral in (13) converges absolutely and uniformly with respect to  $x$  in the interval  $0 \leq x < \infty$ .

**Theorem 4.** Suppose the conditions of Theorem 3 are satisfied and

$$g(x) \in M, \quad h(x) \in L(R^+).$$

Then the following analogue of Parseval' s equality holds:

$$\int_0^\infty g(x) h(x) dx = \sum_{k=1}^r \frac{\alpha_k \beta_k}{\int_0^\infty [y_k(t)]^2 dt} - \frac{1}{2\pi} \int_0^\infty \frac{\alpha(s) \beta(s)}{A(s) \tilde{A}(s)} ds, \quad (14)$$

where

$$\alpha_k = \int_0^\infty g(t) y_k(t) dt, \quad \alpha(s) = \int_0^\infty g(t) \tilde{y}(t, s) dt,$$

$$\beta_k = \int_0^\infty h(t)y_k(t) dt, \quad \beta(s) = \int_0^\infty h(t)\tilde{y}(t, s) dt.$$

Here the integrals on the left- and right-hand sides of (14) converge absolutely.

Suppose that the function  $A(s)$  has multiple zeros. Let  $\lambda_1, \lambda_2, \dots, \lambda_r$  be all the eigenvalues of the operator  $L_{\theta(\lambda)}$ , and let  $s_1, s_2, \dots, s_r$  be the zeros of the function

$$A(s) = y_1'(0, s) - \theta(\lambda)y_1(0, s)$$

of multiplicities, respectively,  $m_1, m_2, \dots, m_r$ . Denote by  $y_{0,k}(x), y_{1,k}(x), \dots, y_{m_k-1,k}(x)$  chains of eigenfunctions and associated functions corresponding to the eigenvalues  $\lambda_k, k = 1, 2, \dots, r$ . Then the following holds.

**Theorem 5.** Suppose  $x^2p(x) \in L(R^+)$ , the functions  $A(s), \tilde{A}(s)$  are nonzero on the half-axis  $s \geq 0$ , and  $\theta(\lambda)$  is the same as in Theorem 2. Then, for any  $\lambda \in G$  not belonging to the spectrum of the operator  $\theta(\lambda)$ ,

$$K(x, t, \lambda) = \sum_{k=1}^r \sum_{q=1}^{m_k} \frac{\alpha_{q,k}(x, t)}{(\lambda - \lambda_k)^q} - \frac{1}{2\pi} \int_0^\infty \frac{\tilde{y}(x, s)\tilde{y}(t, s)}{(s^2 - \lambda)A(s)\tilde{A}(s)} ds, \quad (15)$$

where

$$\alpha_{q,k}(x, t) = \sum_{j=0}^{m_k-q} C_{j+q,k} \sum_{\nu=0}^j y_{\nu,k}(x)y_{j-\nu,k}(t),$$

$C_{q,k}$  are constants, which are determined uniquely, and the integral on the right-hand side of (15) converges absolutely and uniformly with respect to the set of variables  $x, t$  in the domain  $0 \leq x, t < \infty$ .

Let us note that analogous results are also obtained for a certain class of operators of higher order.

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## CITED LITERATURE

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<sup>2</sup> A. V. Shtraus, *Izv. AN SSSR, ser. matem.*, **20**, 783 (1956).

*Note: Figure translations are in progress. See original paper for figures.*

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