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Abstract

Full Text

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ON THE EINSTEIN TENSOR OF FOURTH RANK

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Let us introduce two fourth-rank tensors $\Pi_{\mu\alpha,\beta\nu}$ and $E_{\mu\alpha,\beta\nu}$, setting

$$\Pi_{\mu\alpha,\beta\nu} = g_{\mu\nu}R_{\alpha\beta} + g_{\alpha\beta}R_{\mu\nu} - g_{\nu\alpha}R_{\mu\beta} - g_{\mu\beta}R_{\nu\alpha} - \frac{1}{2}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\nu\alpha}g_{\mu\beta}\right)R; \quad (1)$$

$$R_{\mu\alpha,\beta\nu} = E_{\mu\alpha,\beta\nu} + \Pi_{\mu\alpha,\beta\nu}, \quad (2)$$

where, as usual,* $g_{\mu\nu}$ is the fundamental tensor, $R_{\mu\alpha,\beta\nu}$ is the fourth-rank curvature tensor, $R_{\mu\nu}$ is the second-rank curvature tensor, and R is the scalar curvature.

We note that the tensor $\Pi_{\mu\alpha,\beta\nu}$ differs from analogous tensors given in works ($^{1-5}$) only by the choice of coefficients and, in essence, is a special case of the tensor indicated in work (^{4}).

Let us consider the properties of the tensors $\Pi_{\mu\alpha,\beta\nu}$ and $E_{\mu\alpha,\beta\nu}$. With respect to the tensor $\Pi_{\mu\alpha,\beta\nu}$ the following assertions hold:

1. The tensor $\Pi_{\mu\alpha,\beta\nu}$ possesses the symmetry properties of the Riemann tensor $R_{\mu\alpha,\beta\nu}$.
2. For the divergence of the tensor $\Pi_{\mu\alpha,\beta\nu}$ the equalities hold

$$g^{\alpha\sigma}\nabla_{\sigma}\Pi_{\mu\alpha,\beta\nu} = \nabla_{\beta}R_{\mu\nu} - \nabla_{\nu}R_{\mu\beta}; \quad (3)$$

$$g^{\alpha\sigma}\nabla_{\sigma}\Pi_{\mu\alpha,\beta\nu} = g^{\alpha\sigma}\nabla_{\sigma}R_{\mu\alpha,\beta\nu}. \quad (4)$$

3. If we set

$$\Pi_{\mu\nu} = g^{\alpha\beta}\Pi_{\mu\alpha,\beta\nu}, \qquad \Pi = g^{\mu\nu}\Pi_{\mu\nu},$$

then, by virtue of the definition of the tensor $\Pi_{\mu\alpha,\beta\nu}$,

$$\Pi_{\mu\nu} = 2\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right), \qquad \Pi = 0. \quad (5)$$

4. For the divergence of the tensor $\Pi_{\mu\nu}$ the equalities are valid

$$g^{\mu\sigma}\nabla_{\sigma}\Pi_{\mu\nu} = \frac{1}{2}\frac{\partial R}{\partial x_{\nu}}; \tag{6}$$

$$g^{\mu\sigma}\nabla_{\sigma}\Pi_{\mu\nu} = g^{\mu\sigma}\nabla_{\sigma}R_{\mu\nu}. \tag{7}$$

5. Not only is the tensor $\Pi_{\mu\nu}$ expressed through the tensor $\Pi_{\mu\alpha,\beta\nu}$, but, conversely, the tensor $\Pi_{\mu\alpha,\beta\nu}$ is expressed through the tensor $\Pi_{\mu\nu}$, namely

$$\Pi_{\mu\alpha,\beta\nu} = \frac{1}{2}\left(g_{\mu\nu}\Pi_{\alpha\beta} + g_{\alpha\beta}\Pi_{\mu\nu} - g_{\nu\alpha}\Pi_{\mu\beta} - g_{\mu\beta}\Pi_{\nu\alpha}\right). \tag{8}$$

* Greek indices take the values $0, 1, 2, 3$. Summation from 0 to 3 is assumed over identical Greek indices.

Equations

$$\Pi_{\mu\alpha,\beta\nu} = 0 \tag{9}$$

are equivalent to the equations

$$\Pi_{\mu\nu} = 0 \tag{9*}$$

and determine a space of constant curvature.

Thus, with respect to a four-dimensional space-time of constant curvature, the tensor $\Pi_{\mu\alpha,\beta\nu}$ plays the same role as the Riemann tensor $R_{il,mk}$ with respect to a three-dimensional* Euclidean space.

Let us proceed to consider the properties of the tensor $E_{\mu\alpha,\beta\nu}$. The following assertions hold:

1. The tensor $E_{\mu\alpha,\beta\nu}$ possesses the symmetry properties of the Riemann tensor $R_{\mu\alpha,\beta\nu}$.
2. The tensor $E_{\mu\alpha,\beta\nu}$ is conservative, i.e.

$$g^{\alpha\sigma}\nabla_{\sigma}E_{\mu\alpha,\beta\nu} = 0. \tag{10}$$

3. If one sets

$$E_{\mu\nu} = g^{\alpha\beta}E_{\mu\alpha,\beta\nu}, \qquad E = g^{\mu\nu}E_{\mu\nu},$$

then, by virtue of the definition of the tensor $E_{\mu\alpha,\beta\nu}$,

$$E_{\mu\nu} = -G_{\mu\nu}, \qquad E = R. \tag{11}$$

Here

$$G_{\mu\nu} = R_{\mu\nu} - {}^1/_2 g_{\mu\nu} R \tag{12}$$

is the Einstein tensor.

Consequently, Einstein' s gravitational equations can be written in the form

$$E_{\mu\nu} = \chi T_{\mu\nu}, \tag{13}$$

where χ is Einstein' s gravitational constant and $T_{\mu\nu}$ is the mass tensor.

We shall call the tensor $E_{\mu\alpha,\beta\nu}$ the Einstein tensor of the fourth rank. In formulating certain properties of the Einstein tensor of the fourth rank it is convenient to use the quantities

$$A^{\mu\nu\alpha\beta\sigma\tau} = g^{\mu\nu} g^{\alpha\beta} g^{\sigma\tau} - g^{\mu\alpha} g^{\beta\nu} g^{\sigma\tau} - g^{\mu\tau} g^{\nu\sigma} g^{\alpha\beta}; \tag{14}$$

$$B^{\sigma\tau}_{\mu\nu\alpha\beta} = {}^1/_2 \left(\Gamma^{\sigma}_{\mu\nu} \Gamma^{\tau}_{\alpha\beta} - \Gamma^{\sigma}_{\mu\alpha} \Gamma^{\tau}_{\beta\nu} \right), \tag{15}$$

where $\Gamma^{\alpha}_{\mu\nu}$ are Christoffel symbols of the second kind.

From the definitions (14) and (15) it follows that

$$A^{\mu\nu\alpha\beta\sigma\tau} = A^{\mu\nu\tau\sigma\beta\alpha}, \quad A^{\mu\nu\alpha\beta\sigma\tau} = A^{\mu\nu\beta\alpha\tau\sigma}, \quad A^{\mu\nu\alpha\beta\sigma\tau} + A^{\beta\nu\alpha\mu\sigma\tau} = A^{\mu\nu\alpha\sigma\beta\tau} + A^{\beta\nu\alpha\sigma\mu\tau}; \tag{16}$$

$$B^{\sigma\tau}_{\mu\nu\alpha\beta} = -B^{\sigma\tau}_{\mu\alpha\nu\beta}, \quad B^{\sigma\tau}_{\mu\nu\alpha\beta} = -B^{\tau\sigma}_{\beta\nu\alpha\mu}, \quad B^{\sigma\tau}_{\mu\nu\alpha\beta} + B^{\sigma\tau}_{\mu\alpha\beta\nu} + B^{\sigma\tau}_{\mu\beta\nu\alpha} = 0. \tag{17}$$

Let us continue the consideration of the properties of the Einstein tensor of the fourth rank.

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* Latin indices take the values $1, 2, 3$.

If we put

$$E^{\mu\alpha,\beta\nu} = g^{\mu\varrho} g^{\alpha\sigma} g^{\beta\gamma} g^{\nu\tau} E_{\varrho\sigma,\gamma\tau},$$

$$\begin{aligned} U^{\mu\alpha,\beta\nu} = {} & (A^{\mu\sigma\nu\alpha\tau\beta} - A^{\mu\sigma\nu\tau\alpha\beta})(B^{\varrho\gamma}_{\varrho\sigma\tau\gamma} + B^{\varrho\gamma}_{\gamma\tau\sigma\varrho}) + \\ & + A^{\mu\sigma\nu\tau\beta}(B^{\alpha\gamma}_{\varrho\sigma\gamma\tau} + B^{\alpha\gamma}_{\tau\gamma\sigma\varrho}) - A^{\alpha\sigma\nu\tau\beta}(B^{\mu\gamma}_{\varrho\sigma\gamma\tau} + B^{\mu\gamma}_{\tau\gamma\sigma\varrho}) + \\ & + A^{\nu\sigma\mu\tau\alpha}(B^{\beta\gamma}_{\varrho\sigma\gamma\tau} + B^{\beta\gamma}_{\tau\gamma\sigma\varrho}) - A^{\beta\sigma\mu\tau\alpha}(B^{\nu\gamma}_{\varrho\sigma\gamma\tau} + B^{\nu\gamma}_{\tau\gamma\sigma\varrho}) + \\ & + A^{\mu\nu\sigma\tau\varrho\gamma}B^{\alpha\beta}_{\sigma\varrho\tau\gamma} - A^{\alpha\nu\sigma\tau\varrho\gamma}B^{\mu\beta}_{\sigma\varrho\tau\gamma} + A^{\alpha\beta\sigma\tau\varrho\gamma}B^{\mu\nu}_{\sigma\varrho\tau\gamma} - A^{\mu\beta\sigma\tau\varrho\gamma}B^{\alpha\nu}_{\sigma\varrho\tau\gamma}, \end{aligned} \tag{18}$$

then it can be shown that

$$g(E^{\mu\alpha,\beta\nu} + U^{\mu\alpha,\beta\nu}) = \frac{1}{2}\frac{\partial^2}{\partial x_\sigma \partial x_\tau}\, g(A^{\mu\sigma\nu\alpha\tau\beta} - A^{\mu\sigma\nu\tau\alpha\beta}). \tag{19}$$

Here g is the determinant formed from the components of the fundamental tensor $g_{\mu\nu}$.

From the equalities (19) the conservation laws follow directly:

$$\frac{\partial}{\partial x_\alpha}\, g(E^{\mu\alpha,\beta\nu} + U^{\mu\alpha,\beta\nu}) = 0. \tag{20}$$

Along with the conservation laws (20), others may also be indicated. Let

$$\delta^{\alpha\beta}_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu, \tag{21}$$

where $\delta^\nu_\mu = 1$ for $\mu = \nu$ and $\delta^\nu_\mu = 0$ for $\mu \neq \nu$. From the definitions (21) it follows that

$$\delta^{\alpha\beta}_{\mu\nu} = -\delta^{\beta\alpha}_{\mu\nu}, \qquad \delta^{\alpha\beta}_{\mu\nu} = -\delta^{\alpha\beta}_{\nu\mu}.$$

Now putting

$$E^{\alpha\beta}_{\mu\nu} = g^{\alpha\sigma} g^{\beta\tau} E_{\mu\nu,\sigma\tau},$$

$$\begin{aligned} w^{\alpha\beta}_{\mu\nu\sigma\tau} = {} & \delta^{\alpha\beta}_{\mu\nu} B^{\varrho\gamma}_{\gamma\varrho\sigma\tau} + \delta^{\alpha\beta}_{\mu\sigma} B^{\varrho\gamma}_{\nu\varrho\tau\gamma} - \delta^{\alpha\beta}_{\nu\sigma} B^{\varrho\gamma}_{\mu\varrho\tau\gamma} + \\ & + \delta^{\alpha\beta}_{\mu\gamma} B^{\varrho\gamma}_{\nu\sigma\varrho\tau} - \delta^{\alpha\beta}_{\nu\gamma} B^{\varrho\gamma}_{\mu\sigma\varrho\tau} + \delta^{\alpha\beta}_{\mu\gamma} B^{\varrho\gamma}_{\sigma\nu\varrho\tau} - \delta^{\alpha\beta}_{\nu\gamma} B^{\varrho\gamma}_{\sigma\mu\varrho\tau} - \delta^{\alpha\beta}_{\varrho\sigma} B^{\varrho\gamma}_{\tau\mu\nu\gamma} + 2B^{\alpha\beta}_{\sigma\mu\nu\tau}; \end{aligned} \tag{22}$$

$$V^{\alpha\beta}_{\mu\nu} = g^{\sigma\tau} w^{\alpha\beta}_{\mu\nu\sigma\tau}, \tag{23}$$

one can show that

$$\sqrt{-g}(E^{\alpha\beta}_{\mu\nu} + V^{\alpha\beta}_{\mu\nu}) = \frac{1}{2}\frac{\partial}{\partial x_\sigma}\left\{\frac{g_{\mu\varrho}g_{\nu\gamma}}{\sqrt{-g}}\frac{\partial}{\partial x_\tau}g(A^{\alpha\tau\gamma\sigma\beta\varrho} - A^{\alpha\tau\gamma\beta\sigma\varrho})\right\}. \tag{24}$$

From the equalities (24) the conservation laws obviously follow:

$$\frac{\partial}{\partial x_\alpha}\sqrt{-g}(E^{\alpha\beta}_{\mu\nu} + V^{\alpha\beta}_{\mu\nu}) = 0. \tag{25}$$

Let us show that the Einstein conservation laws follow from (25). Contracting (25) over two indices, we have

$$\frac{\partial}{\partial x_\alpha}\sqrt{-g}(E^{\alpha\sigma}_{\sigma\beta} + V^{\alpha\sigma}_{\sigma\beta}) = 0. \tag{26}$$

In this case

$$E^{\alpha\sigma}_{\sigma\beta} = -\left(R^\alpha_\beta - \frac{1}{2}\delta^\alpha_\beta R\right); \tag{27}$$

$$V^{\alpha\sigma}_{\sigma\beta} = \delta^\alpha_\beta g^{\sigma\tau}B^{\rho\gamma}_{\gamma\sigma\tau} + g^{\alpha\sigma}B^{\rho\gamma}_{\beta\sigma\rho\gamma} + g^{\sigma\tau}B^{\alpha\rho}_{\sigma\tau\rho\beta} + g^{\sigma\tau}B^{\alpha\rho}_{\sigma\beta\rho\tau}. \tag{28}$$

If, as usual, we set

$$L = g^{\sigma\tau}\left(\Gamma^\rho_{\sigma\gamma}\Gamma^\gamma_{\tau\rho} - \Gamma^\rho_{\sigma\tau}\Gamma^\gamma_{\rho\gamma}\right)$$

or, what is the same,

$$L = 2g^{\sigma\tau}B^{\rho\gamma}_{\gamma\sigma\tau}$$

and use the formula

$$\frac{\partial g_{\sigma\tau}}{\partial x_\beta}\frac{\partial L}{\partial\frac{\partial g_{\sigma\tau}}{\partial x_\alpha}} = 2\left(g^{\alpha\sigma}B^{\rho\gamma}_{\beta\rho\sigma\gamma} + g^{\sigma\tau}B^{\alpha\rho}_{\sigma\rho\tau\beta} + g^{\sigma\tau}B^{\alpha\rho}_{\sigma\rho\beta\tau}\right), \tag{29}$$

then the equalities (28) are rewritten in the form

$$V^{\alpha\sigma}_{\sigma\beta} = \frac{1}{2}\left(L\delta^\alpha_\beta - \frac{\partial g_{\sigma\tau}}{\partial x_\beta}\frac{\partial L}{\partial\frac{\partial g_{\sigma\tau}}{\partial x_\alpha}}\right) \tag{30}$$

or

$$V_{\sigma\beta}^{\alpha\sigma} = \chi\theta_{\beta}^{\alpha}, \tag{31}$$

where θ_{β}^{α} is the well-known Einstein pseudotensor of energy-momentum.

Consequently, contracting (25) over two indices, we arrive at the well-known conservation laws

$$\frac{\partial}{\partial x_{\alpha}}\sqrt{-g}\left(T_{\beta}^{\alpha} + \theta_{\beta}^{\alpha}\right) = 0. \tag{32}$$

It follows from the foregoing that the use of the Einstein tensor of the fourth rank makes it possible to formulate conservation laws expressed in terms of quantities of the fourth rank. From these laws there follow in a natural way (by contraction over two indices) the well-known conservation laws expressed in terms of quantities of the second rank.

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Note: Figure translations are in progress. See original paper for figures.

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