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**Abstract**

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# INDECOMPOSABLE MEASURES IN DYNAMICAL SYSTEMS

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Let us consider a dynamical system with compact phase space, i.e., a continuous one-parameter group $\{S_t\}$ of homeomorphisms of a compact set $R$ onto itself. As is known, a normalized measure invariant for the system $\{S_t\}$ is called indecomposable if, for every measurable set invariant with respect to the system, this measure is equal either to 0 or to 1. The indecomposability of a measure with respect to a single homeomorphism $S_t$ is defined analogously.

The following theorem establishes a connection between measures invariant and indecomposable for the whole system $\{S_t\}$ and for its individual homeomorphisms. All measures are assumed to be normalized.

**Theorem 1.** $1^\circ$. For any fixed $t_0 \neq 0$ and any measure $m$, invariant and indecomposable for $\{S_t\}$, there exists a measure $\mu$, invariant and indecomposable for $S_{t_0}$, such that

$$m(A) = \frac{1}{t_0} \int_0^{t_0} \mu(S_{tA})\, dt \tag{1}$$

for any Borel set $A \subset R$.

$2^\circ$. The totality of all measures $\mu$ for which the representation (1) holds for a given measure $m$ is given by the formula

$$\mu_t(A) = \mu(S_{tA}), \tag{2}$$

where $\mu$ is any one of such measures.

$3^\circ$. Every measure $\mu$, invariant and indecomposable for $S_{t_0}$, determines by formula (1) some measure $m$, invariant and indecomposable for the system $\{S_t\}$.

**Proof.** We first prove assertion $3^\circ$. The fact that formula (1) defines some measure in the space $R$ is obvious. Further:

$$m(S_\tau A) = \frac{1}{t_0} \int_0^{t_0} \mu(S_{tS_\tau} A)\, dt = \frac{1}{t_0} \int_0^{t_0} \mu(S_{t+\tau} A)\, dt =$$

$$= \frac{1}{t_0} \int_{\tau}^{t_0+\tau} \mu(S_{tA})\, dt = \frac{1}{t_0} \int_0^{t_0} \mu(S_{tA})\, dt = m(A),$$

since the function $\mu(S_{tA})$ has period $t_0$ by virtue of the invariance of the measure $\mu$ with respect to $S_{t_0}$. This proves the invariance of the measure $m$ with respect to the system $\{S_t\}$. If now $A$ is a set invariant for $\{S_t\}$, then $S_{tA} = A$ for every $t$, and consequently,

$$m(A) = \frac{1}{t_0} \int_0^{t_0} \mu(A)\, dt = \mu(A) = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\},$$

for $\mu$ is indecomposable.

We pass to assertion 1°. Let $m$ be any indecomposable measure in the system $\{S_t\}$, and let $E$ be its ergodic set ([1]). The set $E$, being invariant for the system, is also invariant for $S_{t_0}$. The same is true for the measure $m$. Therefore the measure $m$ is the limit of some sequence of linear combinations of measures indecomposable for $S_{t_0}$. Consequently, there is a measure $\mu$, indecomposable for $S_{t_0}$, such that $\mu(E) = 1$. Then the measure $m'$, defined from the measure $\mu$ by formula (1), will coincide with the measure $m$ on the set $E$, since, obviously, $m'(E) = 1$, and an ergodic set determines only one indecomposable measure. 1° is proved.

Finally, suppose that

$$m(A) = \frac{1}{t_0} \int_0^{t_0} \mu_1(S_{tA})\, dt = \frac{1}{t_0} \int_0^{t_0} \mu_2(S_{tA})\, dt, \tag{3}$$

where $\mu_1, \mu_2$ are measures indecomposable for $S_{t_0}$. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the ergodic sets corresponding to these measures. Put $E_1 = \bigcup_t S_t(\mathcal{E}_1)$, $E_2 = \bigcup_t S_t(\mathcal{E}_2)$. Since $\mu_1(\mathcal{E}_1) = 1$ and $\mu_2(\mathcal{E}_2) = 1$, it follows, by (3), that $m(E_1) = m(E_2) = 1$, and consequently $E_1 \cap E_2 \neq 0$. If $x \in E_1 \cap E_2$, then $x = S_{t_1} x_1 = S_{t_2} x_2$, where $x_1 \in \mathcal{E}_1$ and $x_2 \in \mathcal{E}_2$. Hence $x_2 = S_{t_1 - t_2} x_1$, and this means that $\mathcal{E}_2 = S_t(\mathcal{E}_1)$ and, consequently, $\mu_1(A) = \mu_2(S_{tA})$, where $t = t_1 - t_2$. 2° is proved.

As is known ([2]), if a measure $m$, indecomposable and invariant with respect to the system $\{S_t\}$, is decomposable with respect to some $S_{t_0}$, where $t_0 \neq 0$, then the system $\{S_t\}$ has at least one nonzero eigenfrequency. The following theorem refines this assertion.

**Theorem 2.** *Let $t_0 \neq 0$ be fixed, and let $m$ be any measure invariant and indecomposable for the system $\{S_t\}$. If the measure $m$ is decomposable for $S_{t_0}$, then there is an integer $k \neq 0$ such that $\lambda = 2k\pi/t_0$ is an eigenfrequency of the system $\{S_t\}$. If, on the other hand, the measure $m$ is indecomposable for $S_{t_0}$, then for $k \neq 0$ there are no numbers of the form $2k\pi/t_0$ among the eigenfrequencies of the system $\{S_t\}$.*

**Proof.** If the measure $m$ is decomposable for $S_{t_0}$, then any measure $\mu$, invariant and indecomposable for $S_{t_0}$ and connected with $m$ by formula (1), will be distinct from $m$. The latter means that the measure $\mu$ cannot be invariant for all $S_t$. Denote by $I(\mu)$ the set of all numbers $t$ such that $\mu$ is invariant with respect to $S_t$. $I(\mu)$ is a closed subgroup of the group of all real numbers.

The group property is obvious. Let us prove closedness. If $t_n \in I(\mu)$ and $t_n \to t$, then for any continuous function $f(x)$

$$\int_R f(x)\,d\mu = \int_R f(S_{t_n}x)\,d\mu \to \int_R f(S_{tx})\,d\mu = \int_R f(x)\,d\mu.$$

Consequently, $t \in I(\mu)$.

Since, as was already noted, $I(\mu)$ cannot contain all real numbers, $I(\mu)$ is a set of numbers of the form $n\tau_0$, where $\tau_0 \neq 0$ and $n$ is an integer. In particular, there is an integer $k \neq 0$ such that $t_0 = k\tau_0$. This means that $S_{t_0} = (S_{\tau_0})^k$, and consequently the measure $\mu$ will be indecomposable also for $S_{\tau_0}$. Let $\mathcal{E}$ be the ergodic set of the homeomorphism $S_{\tau_0}$ corresponding to this measure. Put $E = \bigcup_t S_t(\mathcal{E})$. Then, by (1), $m(E) = 1$, since $\mu(\mathcal{E}) = 1$. We now prove that if $S_t(\mathcal{E}) \cap \mathcal{E} \neq 0$, then $t = n\tau_0$, where $n$ is an integer. Let $x_1 \in S_t(\mathcal{E}) \cap \mathcal{E}$. Then $x_1 = S_{tx}0$, where $x_0 \in \mathcal{E}$. Since the individual measures of the points $x_0$ and $x_1$ coincide with the measure $\mu$, for any continuous-

continuous function $f(x)$

$$\int_R f(S_t x)\,d\mu = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(S_t S_\tau x_0)\,d\tau = \lim_{T\to\infty} \frac{1}{T_0}\int_0^T f(S_\tau S_t x_0)\,d\tau =$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T f(S_\tau x_1)\,d\tau = \int_R f(x)\,d\mu.$$

This means that $t \in I(\mu)$ and, consequently, $t = n\tau_0$. Let now $x \in E$. Then, if $x = S_{t_1}x_1 = S_{t_2}x_2$, where $x_1, x_2 \in \mathcal{E}$, then $x_2 = S_{t_1 - t_2}x_1$ and, consequently, $t_1 - t_2 = n\tau_0$. Therefore there exists a number $\alpha(x)$, least in absolute value, such that $x = S_{\alpha(x)}\tilde{x}$, where $\tilde{x} \in \mathcal{E}$. It is easy to verify that

$$\alpha(S_t x) = \alpha(x) + t + n\tau_0, \tag{4}$$

where $n$ is an integer. Finally, put, for $x \in E$,

$$\varphi(x) = e^{i\frac{2\pi\alpha(x)}{\tau_0}}.$$

Then, by virtue of (4),

$$\varphi(S_t x) = e^{i\lambda t}\varphi(x), \qquad \text{where } \lambda = \frac{2\pi}{\tau_0} = \frac{2k\pi}{t_0}.$$

Thus, $\varphi(x)$ is an eigenfunction of the system $\{S_t\}$ with eigenfrequency $\lambda = 2k\pi/t_0$.

Let now the measure $m$ be indecomposable for $S_{t_0}$. Suppose that there exists an eigenfunction $\varphi(x)$ of the system $\{S_t\}$ with eigenfrequency $\lambda = 2k\pi/t_0$, where $k \neq 0$ is an integer. Put $\tau_0 = t_0/k$. Then $S_{t_0} = (S_{\tau_0})^k$ and, consequently, $m$ will also be indecomposable for $S_{\tau_0}$. Since, obviously, $\varphi(S_{\tau_0}x) = \varphi(x)$, it follows that $\varphi(x)$ is constant almost everywhere (with respect to the measure $m$), which is possible only when $k = 0$. Theorem 2 is proved.

The following theorem is also adjacent to Theorem 2.

**Theorem 3.** *If, for some $t_0 \neq 0$, the homeomorphism $S_{t_0}$ has a nonconstant and measurable $(m)$ eigenfunction with eigenfrequency $\alpha$, then either this function is also an eigenfunction for the system $\{S_t\}$, with frequency $\lambda = \alpha + 2k\pi/t_0$, where $k$ is an integer, or the system $\{S_t\}$ will have an eigenfrequency $\lambda = 2k\pi/t_0$ with an integer $k \neq 0$.*

**Proof.** Let $\varphi(x)$ be an eigenfunction of $S_{t_0}$ with frequency $\alpha$. Then almost everywhere with respect to the measure $m$

$$\varphi(S_{t_0}x) = e^{i\alpha t_0}\varphi(x). \tag{5}$$

Assume first that the measure $m$ is indecomposable for $S_{t_0}$. Since it follows from (5) that $|\varphi(S_{t_0}x)| = |\varphi(x)|$, i.e. $|\varphi(x)|$ is a function invariant with respect to $S_{t_0}$, almost everywhere $|\varphi(x)| = \text{const}$, and one may assume that $|\varphi(x)| = 1$. Further one may assume that (5) holds for all $x \in R$. Then, for arbitrary $t$,

$$\varphi(S_t S_{t_0} x) = \varphi(S_{t_0} S_t x) = e^{i\alpha t_0}\varphi(S_t x). \tag{6}$$

From (5) and (6) it follows that the function $\varphi(S_t x)/\varphi(x)$ is invariant with respect to $S_{t_0}$, and therefore, for every $t$, almost everywhere

$$\varphi(S_t x) = z(t)\varphi(x), \tag{7}$$

where $z(t)$ does not depend on $x$. From (7) it is easy to obtain that for any $t_1$ and $t_2$

$z(t_1 + t_2) = z(t_1)z(t_2)$. Let us note further that equality (7) in the space $L^2_m(R)$ is equivalent to

$$U_t\varphi = z(t)\varphi, \tag{8}$$

where $U_t$ is the unitary operator corresponding to $S_t$. It follows from (8) that $(U_t\varphi, \varphi) = z(t)(\varphi, \varphi)$, and, consequently, $z(t)$ is a continuous function of $t$. Finally, since $|\varphi(x)| = 1$, we have $|z(t)| = 1$, and therefore $z(t) = e^{i\lambda t}$, where $\lambda$ is a real number. Thus (7) takes the form

$$\varphi(S_t x) = e^{i\lambda t}\varphi(x), \tag{9}$$

and comparison of (9) and (5) shows that $\lambda = \alpha + 2k\pi/t_0$. The first assertion of the theorem is proved. As for the second, it follows directly from Theorem 2 if the measure $m$ is decomposable for $S_{t_0}$.

**Remark.** If the system $\{S_t\}$ is strictly ergodic, i.e., admits a unique invariant normalized measure, then Theorem 1 of Note [3] follows in an obvious way from Theorem 2.

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## REFERENCES

[1] V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow–Leningrad, 1949, pp. 529–531. [2] V. A. Rokhlin, *Uspekhi Mat. Nauk*, **4**, no. 2 (30) (1949). [3] M. I. Grabar' , *Dokl. Akad. Nauk SSSR*, **95**, no. 1 (1954).

*Note: Figure translations are in progress. See original paper for figures.*