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Abstract

Full Text

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On Dyadic Bicompacta

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§ 1. Auxiliary propositions.

Let

$$D^{\tau} = \prod_{\alpha \in \theta} D_{\alpha}^{0,1}, \qquad \operatorname{card} \theta = \tau,$$

where $D_{\alpha}^{0,1}$ is a space consisting of two isolated points, and $\prod$ denotes topological product. Let $w \subset \theta$. We shall call the set $H_w^{i(w)}$ of those points $\{x_\alpha\} \in D^{\tau}$ for which, for all $\alpha \in w$, we have $x_\alpha = i_\alpha \in i(w)$, while the remaining coordinates are arbitrary, a *layer* of the space D^{τ} with base w. Let $F \subset D^{\tau}$ be closed, and let $\chi F = \tau_0 \leqslant \tau$ be the least cardinal number for which there exists a system $\{H_\nu\}$ of cardinality τ_0 of open-and-closed sets with intersection $\bigcap_\nu H_\nu = F$. This number χF will be called the *neighborhood character* of the set F (in the space D^{τ}).

It can be proved that each open-and-closed set H_ν is the sum of a finite number of elementary neighborhoods with one and the same base; therefore

$$F = \bigcap_{\nu} \left(\bigcup_{s=1}^{k(\nu)} H_{\lambda_\nu}^{s} \right) = \bigcup_{\xi} \bigcap_{\nu, s} H_{\lambda_\nu}^{s}, \qquad 1 \leqslant s \leqslant k(\nu). \tag{1}$$

Here by ξ are meant all possible nonempty intersections formed as follows: from each open-and-closed H_ν an arbitrary neighborhood $H_{\lambda_\nu}^{s}$, $1 \leqslant s \leqslant k(\nu)$, is taken, and the intersection of all selected neighborhoods is considered. Since an elementary neighborhood is the intersection of a finite number of one-index neighborhoods, we have

$$\bigcap_{\nu, s} H_{\lambda_\nu}^{s} = \bigcap_{\nu} \left(H_{\alpha_{1\nu}}^{i_{1\nu}} \cap \dots \cap H_{\alpha_{t(s),\nu}}^{i_{t(s),\nu}} \right) = \bigcap_{\mu} H_{\alpha_\mu}^{i_\mu} = H_w^{i(\xi)}, \tag{2}$$

where $w = \bigcup_\mu \alpha_\mu$, and $i(\xi)$ is the corresponding collection of zeros and ones, depending on ξ. Thus $H_w^{i(\xi)}$ will be a layer of D^{τ} with base w, where $\operatorname{card} w \leqslant \tau_0$. Combining (1) and (2), we obtain that $F = \bigcup_\xi H_w^{i(\xi)}$, with $\operatorname{card} w \leqslant \chi F$.

Every decomposition of a closed set F into $\bigcup_\xi H^{i(\xi)}_{w(\xi)}$ will be called a *stratification* of F.

We have now proved the following proposition:

A. If $\chi F = \tau_0$, then there exists a stratification of F, the cardinality of the base of each layer of which does not exceed the neighborhood character of F.

It can be shown that if the base w is one and the same for each layer, then the stratification is a continuous partition of the set F. The space of this partition will be called the *skeleton* of F ($\operatorname{sk} F$). In this case we have the following two properties of $\operatorname{sk} F$.

B. The weight of the skeleton does not exceed the neighborhood character of F.

C. If at least one skeleton of the bicompactum F is dyadic, then the bicompactum F itself is dyadic (and then each of its skeletons is dyadic).

We now prove the following proposition.

D. If $F = \left[\bigcup_\xi H^{i(\xi)}_{w(\xi)}\right]$, where $\operatorname{card} w_\xi \leqslant \tau_0$, then $\chi F \leqslant \tau_0$.

Proof. We may suppose that $D^\tau \setminus F \neq \Lambda$. For any point $x \in D^\tau \setminus F$ consider

$$H_x = H^{i_1 \dots i_s}_{a_1 \dots a_s} \subset D^\tau \setminus F.$$

Since

$$H^{i_1 \dots i_s}_{a_1 \dots a_s} \cap H^{i(\xi)}_w(\xi) = \Lambda$$

for any ξ, this means that for each ξ we have

$$e = a_1, \dots, a_s \cap w(\xi) \neq \Lambda$$

and there is at least one $a_j = a^\xi_\mu \in e$ such that

$$i_{a_j} \neq i_{a^\xi_\mu}.$$

Call this condition (L). A neighborhood

$$H^{i_1 \dots i_s}_{a_1 \dots a_s}$$

satisfying condition (L) will be called **regular** if there is no proper part $a_1, \dots, a_k \subset a_1, \dots, a_s$ such that the neighborhood

$$H^{i_1 \dots i_k}_{a_1 \dots a_k}$$

satisfies condition (L). It can be shown that every point $x \in D^\tau \setminus F$ has a regular neighborhood. If we show that the cardinality of the set of distinct regular neighborhoods does not exceed τ_0, then it will thereby be proved that

$$\chi F \leq \tau_0.$$

Denote this family by

$$H = \{H^{i_\mu}_{\lambda_\mu}\}.$$

Suppose the contrary. Let $\tau_0 = \aleph_\sigma$; then $\operatorname{card} H \geq \aleph_{\sigma+1}$ is uncountable. Therefore there exists a subfamily of the family H (which we shall still denote by

$$H = \{H^{i_\mu}_{\lambda_\mu}\}),$$

all neighborhoods of which have one and the same rank s, and moreover

$$\operatorname{card} H \geq \aleph_{\sigma+1}.$$

Consider some layer

$$H^{i(\xi_0)}_{w(\xi_0)} \subset F.$$

From condition (L) it follows that for each neighborhood $H^{i_\mu}_{\lambda_\mu} \in H$,

$$\lambda_\mu \cap w(\xi_0) \neq \Lambda.$$

To each

$$a^\xi_\nu \in w(\xi_0)$$

assign the cardinality of those $H^{i_\mu}_{\lambda_\mu} \in H$ whose bases contain the index a^ξ_ν. Since

$$\operatorname{card} w(\xi) \leq \aleph_\sigma,$$

and

$$\operatorname{card} H \geq \aleph_{\sigma+1},$$

there exists a subfamily H_1 of the family H, with

$$\operatorname{card} H_1 \geq \aleph_{\sigma+1},$$

all bases of whose neighborhoods contain one and the same index a_1, and the a_1-st coordinate is the same in all neighborhoods. Next we continue by induction. Suppose that we have found a subfamily H^{n-1} of the family H, with

$$\operatorname{card} H^{n-1} \geq \aleph_{\sigma+1},$$

all bases of whose neighborhoods contain the same indices $a_1, \ldots, a_{n-1}$, and the values

$$i_{a_1}, \ldots, i_{a_{n-1}}$$

coincide in all neighborhoods of the family H^{n-1}. Consider an arbitrary representative of this family

$$H_\lambda = H^{i_{a_1} \ldots i_{a_{n-1}}\, i_{a_n} \ldots i_{a_s}}_{a_1 \ldots a_{n-1}\, a_n \ldots a_s}.$$

The neighborhood

$$H^{i_{a_1}\dots i_{a_{n-1}}\, i_{a_{n+1}}\dots i_{a_s}}_{a_1\dots a_{n-1}\, a_{n+1}\dots a_s},$$

which is obtained from H_λ by omitting the index a_n, is no longer regular; consequently, there exists a layer

$$H^{i(\xi_0)}_{w(\xi_0)} \subset F,$$

for which either 1)

$$a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_s \cap w(\xi_0) = \Lambda$$

or 2)

$$a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_s \cap w(\xi_0) \neq \Lambda$$

and for all $a_j = a_\nu^{\xi_0}$ we have

$$i_{a_j} = i_{a_\nu^{\xi_0}}, \qquad a_\nu^{\xi_0} \in w(\xi_0).$$

In case 1), every λ_μ intersects $w(\xi_0)$ in indices not belonging to $a_1, \dots, a_{n-1}$. Since

$$\operatorname{card} w(\xi_0) \leq \aleph_\sigma$$

and

$$\operatorname{card} H^{n-1} \geq \aleph_{\sigma+1},$$

there exists a subfamily H^n of the family H^{n-1}, with

$$\operatorname{card} H^n \geq \aleph_{\sigma+1},$$

all bases of whose neighborhoods contain one and the same index

$$a_\nu^{\xi_0} = a_n$$

and for which i_{a_n} coincides. In case 2), we again find that every λ_μ intersects $w(\xi_0)$ in indices not belonging to $a_1, \dots, a_{n-1}$, since every neighborhood

$$H^{i_\mu}_{\lambda_\mu} \in H^{n-1}$$

satisfies condition (L). Thus, in both cases we obtain that in the family H^n all bases of neighborhoods contain the same indices

$$a_1, \dots, a_n,$$

and all values

$$i_{a_1}, \dots, i_{a_n}$$

coincide in all neighborhoods. Consequently, after no more than s steps we obtain that there exists a subfamily H^s of the family H, with

$$\operatorname{card} H^s \geq \aleph_{\sigma+1},$$

all neighborhoods of which coincide, which contradicts the fact that all

$$H^{i_\mu}_{\lambda_\mu}$$

are distinct. D is proved.

The following propositions easily follow from D.

Theorem 1. A canonical closed set in the space D^τ is a set of type G_δ (consequently, every canonical open set is a set of type F_σ).

Next we have the proposition:

If the character of a closed set F is uncountable, then in any partition of F there exists a layer $H^i_w(\xi_0)$ such that, for every countable subset $v \subset w(\xi_0)$, the layer H^i_v, containing $H^i_{w(\xi_0)}(\xi_0)$, intersects $D^\tau \setminus F$.

Now the following known theorem is easily proved:

Theorem 2. The weight of a dyadic bicompactum R is equal to the least upper bound of the values of the neighborhood characters of the points $x \in R$.

Indeed, let

$$\tau_0 = \sup_{x \in R} \chi x$$

and $R = f(D^\tau)$; then, to prove Theorem 2, it is enough to show that $\chi_{R\times R}\Delta \le \tau_0$. It is easy to note that $\chi y \le \tau_0$ for all $y \in \Delta$ and that $R \times R = g(D^\tau)$. Then

$$\chi_{D^\tau}(g^{-1}\Delta) = \chi_{D^\tau}\left(\bigcup_{y\in\Delta} F_y\right),$$

where $F_y = g^{-1}y$. Since $\chi F_y \le \tau_0$, it follows, applying A, that

$$F_y = \bigcup_\xi H^i_{w(y)}(\xi),$$

where $\operatorname{card} w(y) \le \tau_0$; further,

$$g^{-1}\Delta = \bigcup_y F_y = \bigcup_y \bigcup_\xi H^i_{w(y)}(\xi);$$

applying D, we obtain

$$\chi_{D^\tau}(g^{-1}\Delta) \le \tau_0,$$

as was required.

§ 2. **Main results.**

Theorem 3 (the first main theorem). In a dyadic bicompactum every canonical closed set and every closed G_δ are dyadic bicompacta.

Proof. Let $R = f(D^\tau)$; $F = [U] \subset R$, U open. Then

$$F = f(\widetilde{F}),$$

where

$$\widetilde{F} = [f^{-1}U]$$

is canonical closed, which is G_δ, by Theorem 1. If, on the other hand, F is a closed G_δ in R, then its full preimage will be a closed G_δ in D^τ. Thus, in both cases, the proof reduces to establishing the fact that every closed G_δ set $\widetilde{F}$ in D^τ is dyadic. We show this. Since $\widetilde{F}$ is a closed G_δ in D^τ, we have $\chi\widetilde{F} \leq \aleph_0$; applying B, we obtain that the weight $\operatorname{sk} F \leq \aleph_0$; hence, by the theorem of P. S. Aleksandrov ([1]), the bicompactum $\operatorname{sk}\widetilde{F}$ is dyadic; using proposition C, we obtain that $\widetilde{F}$ itself is dyadic. The theorem is proved.

Theorem 4 (the second main theorem). A hereditarily dyadic bicompactum is metrizable.

Proof. Denote by E_τ the space of cardinality $\tau \geq \aleph_0$ consisting of isolated points only, and by bE_τ some bicompact extension of E_τ. In particular, denote by b_0E_τ the bicompactum obtained by adjoining to E_τ a single point ξ, called the vertex of b_0E_τ. If $\tau \geq \aleph_1$, then the bicompactum bE_τ is not dyadic ([2]). We first prove the following proposition:

E. A continuous dyadic image of bE_τ, under the condition that the remainder

$$N = bE_\tau \setminus E_\tau$$

is mapped to a single point, is the bicompactum $b_0E_{\aleph_0}$.

Let $Y = f(bE_\tau)$ and $z = f(N) \in Y$. We show that $Y = b_0E_{\tau'}$, $\tau' \leq \tau$, and z is the vertex of Y. Indeed, let Oz be an arbitrary neighborhood of the point z; choose ON so that

$$f(ON) \subseteq Oz;$$

since outside ON there lies a finite number of isolated points of bE_τ, outside Oz there can lie only the images of these points. Hence it follows at once that z is the unique non-isolated point of the bicompactum Y. Thus

$$Y = b_0E_{\tau'},$$

with $\tau' \leq \tau$. If, moreover, Y is a dyadic bicompactum, then

$$\tau' = \aleph_0.$$

As a consequence of proposition E we obtain that the full preimage of the point z contains all points of E_τ, with the exception of at most countably many. Now let

$$R = f(D^\tau)$$

be a hereditarily dyadic bicompactum. To prove that R is metrizable, it is enough, by Theorem 2, to show that R satisfies the first axiom of countability, i.e. that for each

of the point $x \in R$, $F = f^{-1}x$ has countable neighborhood character in D^τ. Suppose that for some point $x_0 \in R$, $\chi F_0 \geq \aleph_1$, where $F_0 = f^{-1}x_0$. We shall then show that in D^τ there lies a bicompactum $bE_{\aleph_1}$ such that

$$N = bE_{\aleph_1} \setminus E_{\aleph_1} \subset F_0,$$

and moreover $E_{\aleph_1} \cap F_0 = \Lambda$. This means, as was shown above, that $f(bE_{\aleph_1}) = b_0E_{\aleph_1}$, which lies in R, contradicting the hereditary dyadicity of R. We shall construct $bE_{\aleph_1}$ by transfinite induction. Consider some partition F_0. Using the proposition, consider a layer

$$U = H^{i(\xi_0)}_{\mathfrak{w}(\xi_0)}$$

such that, for any countable subset $v \subset \mathfrak{w}(\xi_0)$, the layer $H^{i(v)}_v$ containing U intersects $D^\tau \setminus F_0$. Let $\alpha_1 \in \mathfrak{w}(\xi_0)$. Take $H^{i_{\alpha_1}}_{\alpha_1} \supset U$. There will be found a point $y_1 \in H^{i_{\alpha_1}}_{\alpha_1}$ not belonging to F_0. Consider

$$H_{\lambda_1} = H^{i_1 \ldots i_{s(1)}}_{\beta_1 \ldots \beta_{s(1)}}(y_1) \subset D^\tau \setminus F_0.$$

From H_{λ_1} take a point x_1 such that on the indices $\mathfrak{w}(\xi_0) \setminus \lambda_1$ it assumes the very same values as the layer U, and on all the remaining, non-fixed ones, zeros. Denote by A_1 the set of those indices $\beta \in \lambda_1$ which enter into $\mathfrak{w}(\xi_0)$ and carry opposite values. Such indices necessarily exist. Suppose that, for an arbitrary countable ordinal v, the points $x_1, \ldots, x_\mu, \ldots$, $\mu < v$, and the sets A_μ have been constructed. Denote

$$A^*_v = \bigcup_{\mu < v} A_\mu.$$

It is easy to see that card $A^*_v \leq \aleph_0$, and that $A^*_v \subset \mathfrak{w}(\xi_0)$; therefore, for the layer

$$H^{i(A^*_v)}_{A^*_v} \supset U$$

there will be found a point $y_v \in D^\tau \setminus F_0$. Consider

$$H_{\lambda_v} = H^{i_1 \ldots i_{s(v)}}_{\beta^v_1 \ldots \beta^v_{s(v)}}(y_v) \subset D^\tau \setminus F_0.$$

From H_{λ_v} take a point x_v such that on the indices $\mathfrak{w}(\xi_0) \setminus \lambda_v$ it assumes the very same values as the layer U, and on all the remaining non-fixed ones, zeros. Just as at the first step, from λ_v select the set A_v. The induction is complete. We shall prove that $\{x_v\} = bE_{\aleph_1}$ is the required bicompactum. Note that $A_v \cap A_\mu = \Lambda$. Indeed, let $\mu < v$; then $A_\mu \subset A^*_v = \bigcup_{\gamma < v} A_\gamma$, and therefore the point x_v on the indices A_μ carries the very same values as the layer U, while on the indices A_v it carries opposite values; consequently, A_v and A_μ do not intersect. It follows from this that the neighborhood

$$H^{i(A_v)}_{A_v}$$

does not contain a single point x_μ, since x_μ, on the indices $\mathfrak{w}(\xi_0)\setminus A_\mu$ containing A_v, carries the same values as the layer U, and hence values opposite to the indices $i(A_v)$. Thus $\{x_v\} = E_{\aleph_1}$. We now show that no point of $D^\tau \setminus U$ is a limit point for $\{x_v\}$. If $y \notin U$, this means that there is an index $\alpha \in \mathfrak{w}(\xi_0)$ such that $y_\alpha \neq i_\alpha$, where i_α is the corresponding value of the layer U. In that case the neighborhood

$$H_\alpha^{y_\alpha}(y)$$

can contain no more than one point of the set $\{x_v\}$, since there exists at most one $A_v \ni \alpha$. Thus the remainder $[\{x_v\}]$ lies entirely in the layer U, and consequently in F_0. The theorem is proved.

A strengthening of Theorem 3 is the following theorem:

Theorem 5. *If $\tau \geq \aleph_0$, then every nonempty canonical closed set of the space D^τ is homeomorphic to the whole D^τ. If $\tau \geq \aleph_1$, then every closed G_δ in D^τ is homeomorphic to the whole D^τ.*

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CITED LITERATURE

1. P. S. **Aleksandrov**, *Introduction to the General Theory of Sets and Functions*, 1958.

2. E. **Marczewski**, *Fund. Math.*, **34**, 127 (1947).

Note: Figure translations are in progress. See original paper for figures.