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Abstract

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ASYMPTOTICS AND EXISTENCE OF A SOLUTION OF THE CAUCHY PROBLEM FOR A FIRST-ORDER DIFFERENTIAL EQUATION WITH A SMALL PARAMETER IN A BANACH SPACE

(Presented by Academician S. L. Sobolev on 21 III 1963)

Consider in a real Banach space E the Cauchy problem for the differential equation

$$\varepsilon \frac{dy}{dt} = F(y, \varepsilon), \qquad 0 < t \leq T, \tag{1}$$

$$y(0) = 0. \tag{2}$$

Here $\varepsilon > 0$ is a small parameter; $F(y, \varepsilon)$ in the sphere $\|y\|^2 + \varepsilon^2 \leq \rho^2$ expands into a uniformly convergent power series in the sense of Fréchet

$$F(y, \varepsilon) = \sum_{i+j \geq 1} F_{ij} \varepsilon^i y^j.$$

Everywhere below it is assumed that $F_{ij} y^j$ for $j \geq 2$ are j-linear bounded operators in E, while the operator $B = -F_{01}$ is a linear, generally speaking unbounded, operator with domain dense in E and range closed in E. With respect to the spectrum of the operator B, the fulfillment of one of the following hypotheses is assumed.

Hypothesis I. Zero is not a point of the spectrum of the operator B. The spectrum B is situated in the right half-plane and is such that there exists a strongly continuous semigroup e^{-Bt}, $t \geq 0$, and moreover the estimate $\|e^{-Bt}\| \leq e^{-\beta t}$ holds, where $\beta = \text{const} > 0$. It is also assumed that the Cauchy problem $dz/dt + Bz = 0$, $t > 0$, $z(0) = 0$ has only the trivial solution. Conditions on the spectrum guaranteeing the fulfillment of these properties are available, for example, in the monograph of E. Hille ([1]).

Hypothesis II. Zero is a simple eigenvalue of the operators B and B^*. The null element φ of the operator B and the null element ψ of the operator B^* can

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be normalized so that $(\psi, \varphi) = 1$. Decompose the space E into the direct sum $E = E^1 + E^{\infty-1}$, where E^1 is the null subspace of the operator B, and $E^{\infty-1}$ is the range of the operator B. On $E^{\infty-1}$ the operator B has a bounded inverse $\Gamma = B^{-1}$. We shall assume that the spectrum of the operator B, considered on $E^{\infty-1}$, is such that for B, considered on $E^{\infty-1}$, Hypothesis I holds.

Below are given sufficient conditions ensuring the existence of a solution of problem (1)–(2) and allowing one to construct the asymptotics of this solution. In this note we are guided by certain ideas from works ([2–6]).

Theorem 1. *Let Hypothesis I be satisfied. Then, for sufficiently small $\varepsilon > 0$, there exists a unique solution $y(t, \varepsilon)$ of problem (1)–(2). The asymptotics holds*

$$y = \sum_{i=1}^{N} \left[y_i + v_i \left(\frac{t}{\varepsilon} \right) \right] \varepsilon^i + O\left(\varepsilon^{N+1}\right).$$

Proof. We shall seek an approximate solution of the equation in the form

$$\bar{y}_N = y_1 \varepsilon + y_2 \varepsilon^2 + \dots + y_N \varepsilon^N. \tag{3}$$

Substituting (3) into (1), we obtain, for the determination of $y_1, y_2, \dots, y_N$, a recurrent sequence of problems

$$By_1 = F_{10},$$

$$By_2 = F_{20} + 2F_{11} y_1 + F_{02} y_1^2 - \frac{dy_1}{dt}, \tag{4}$$

$$\dots\dots\dots\dots\dots\dots$$

Hence $y_1 = \Gamma F_{10}$, etc. Since $\bar{y}_N$ does not, generally speaking, satisfy condition (2), we construct functions of boundary-layer type that eliminate this discrepancy in the fulfillment of the boundary condition. Put in (1) $t = \varepsilon\tau$, $y = \bar{y}_N + \bar{v}_N$, where

$$\bar{v}_N = v_1(\tau)\varepsilon + v_2(\tau)\varepsilon^2 + \dots + v_N(\tau)\varepsilon^N. \tag{5}$$

To determine $v_k(\tau)$ we have the system of problems:

$$\frac{dv_1}{d\tau} + Bv_1 = 0, \qquad v_1\big|_{\tau=0} = -y_1,$$

$$\frac{dv_2}{d\tau} + Bv_2 = 2F_{11} v_1 + 2F_{02} y_1 v_1 + F_{02} v_1^2, \tag{6}$$

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$$v_2\big|_{\tau=0} = -y_2$$

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From the first equation of system (6), $v_1 = -e^{-B\tau} y_1$. Obviously, this is a function of boundary-layer type. Further,

$$v_2 = -e^{-B\tau} y_2 + \int_0^\tau e^{-B(\tau - s)} h_2(s)\, ds,$$

where

$$h_2(s) = 2F_{11} v_1(s) + 2F_{02} y_1 v_1(s) + F_{02} v_1^2(s).$$

We have $\|v_2\| \le (c_1 + c_2\tau) e^{-\beta\tau}$. Here c_1 and c_2 are positive constants. It is established analogously that all $v_k(\tau)$ are functions of boundary-layer type. To prove the existence and uniqueness of a solution of problem (1)–(2), we note that this problem is equivalent to the integral equation

$$y(t) = \frac{1}{\varepsilon} \int_0^t e^{-B(t-s)/\varepsilon} \left[F_{10}\varepsilon + \sum_{i+j\ge 2} F_{ij} \varepsilon^i y^j(s) \right] ds. \tag{7}$$

Consider this equation in the Banach space $\widetilde{E}$ of functions $h(t)$, continuous for $0 \le t \le T$, with values in E and with norm

$$|h| = \sup_{0\le t\le T} \|h(t)\|.$$

Our assertion now follows from the fact that the operator standing on the right-hand side of equation (7), for sufficiently small $\varepsilon > 0$, is a contraction operator in some sphere of the space $\widetilde{E}$. If in (7) we put $y = \bar{y}_N + \bar{v}_N + z$, then it is easily established that $|z| = O(\varepsilon^{N+1})$.

Let hypothesis II be satisfied. Introduce the numbers

$$\alpha_i = (\psi, F_{0i}\varphi^i), \qquad i = 2, 3, \dots, \qquad \beta = (\psi, F_{11}\varphi + F_{02}\varphi\Gamma F_{10}),$$

$$\gamma = (\psi, F_{20} + 2F_{11}\Gamma F_{10} + F_{02}(\Gamma F_{10})^2), \qquad \delta = (\psi, F_{10}).$$

Theorem 2. Let hypothesis II be satisfied, $\delta = 0$, $\beta^2 - \alpha_2\gamma > 0$; then the assertions of Theorem 1 hold. (If $\alpha_2 = 0$, assume $\beta < 0$.)

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Proof. We again seek an approximate solution of equation (1) in the form (3), which leads to system (4). Its first equation is solvable and gives $y_1 = \Gamma F_{10} + c_1(t)\varphi$, where $c_1(t)$ is an arbitrary sufficiently smooth function equal to zero at $t = 0$. The solvability condition for the second equation of system (4) gives

$$\frac{dc_1}{dt} = \alpha_2 c_1^2 + 2\beta c_1 + \gamma, \qquad c_1(0) = 0. \tag{8}$$

The solution of this problem under the conditions of the theorem is easily written explicitly. From the second equation of system (4) we have $y_2 = \Gamma f_2(t) + c_2(t)\varphi$, where $f_2(t)$ is a known function, and $c_2(t)$ is a new arbitrary function of t. To determine $c_2(t)$ we use the solvability condition for the third equation of system (4):

$$\frac{dc_2}{dt} - [2\alpha_2 c_1(t) + 2\beta]c_2(t) = g_2(t), \qquad c_2(0) = 0, \tag{9}$$

where $g_2(t)$ is a known function. The solution of this problem is also written out in explicit form. By induction it is easily established that, in order to determine $c_3, \dots, c_N$, one must each time solve a problem of type (9). Thus, $\bar{y}_N$ has been constructed.

The second iterative process. The function $\bar{y}_N$, generally speaking, does not satisfy condition (2). Indeed, $y_1(0) = \Gamma F_{10}$, etc. We construct boundary-layer functions. Since it is obvious that $y_i(0) \in E^{\infty-1}$, the boundary-layer functions should also be sought in this subspace. Proceeding in the same way as in Theorem 1, we obtain, for determining $v_i(\tau)$, system (6). We solve it in $E^{\infty-1}$. For $v_i(\tau)$ we have the same formulas. Thus, an approximate solution of problem (1)–(2) has been constructed.

We shall prove the existence and uniqueness of a solution of this problem for the case $N = 1$. The general case differs only by more cumbersome computations. Put in (1)

$$y = \varepsilon\left[y_1(t) + v_1\left(\frac{t}{\varepsilon}\right) + z(t,\varepsilon) + c(t,\varepsilon)\varphi\right],$$

where y_1, v_1 are defined above and $z \in E^{\infty-1}$. After division by ε we have

$$\varepsilon\frac{dz}{dt} + Bz - \varepsilon\left[\frac{dc}{dt}\varphi - 2F_{11}\varphi c - 2F_{02}y_1\varphi c\right]$$
$$= \varepsilon f(t) + \varepsilon g(t, v_1, z, c) + \varepsilon H(t, \varepsilon, z, c), \qquad z(0) = c(0) = 0. \tag{10}$$

Here $f(t)$ is a known function, with $(\psi, f) = 0$, for $t \geqslant 0$; g and H also are not difficult to write out, and for $\varepsilon = z = c = 0$, $H = \partial H/\partial z = \partial H/\partial c = 0$. Projecting now (10) onto E^1 and $E^{\infty-1}$, we obtain, after dividing the first equation by ε:

$$c'(t) - [2\alpha_2 c_1(t) + 2\beta]c(t) = (\psi, g(t, v_1, z, c)) + (\psi, H),$$

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$$\varepsilon \frac{dz}{dt} + Bz = \varepsilon G(t, \varepsilon, z, c), \tag{11}$$
$$c(0) = z(0) = 0.$$

Here G is the projection onto $E^{\infty - 1}$ of the element

$$-\frac{dc}{dt}\varphi + 2F_{11}\varphi c + 2F_{02} y_1 \varphi c + f(t) + g + H.$$

We reduce problem (11) to a system of integral equations

$$c(t) = c_1'(t) \int_0^t \frac{1}{c_1'(s)} \left(\psi, g\left(s, v_1\left(\frac{s}{\varepsilon}\right), z(s,\varepsilon), c(s,\varepsilon)\right)\right) ds + c_1'(t) \int_0^t \frac{1}{c_1'(s)} (H, \psi)\, ds,$$

$$z(t) = \int_0^t e^{-B(t-s)/\varepsilon} G(s, \varepsilon, z(s,\varepsilon), c(s,\varepsilon))\, ds. \tag{12}$$

The order of the first term in the right-hand side of the first equation of system (12) and of the right-hand side of the second equation is $O(\varepsilon)$. To see this, it suffices to make the substitution $s = \varepsilon\sigma$ in the integrals.

Theorem 3. *Suppose that Hypothesis II is satisfied, $\delta \neq 0$, $\alpha_2 = \cdots = \alpha_{k-1} = \alpha_k \neq 0$. Suppose that for even k, $\delta\alpha_k < 0$, and for odd k, $\alpha_k < 0$. Then, for sufficiently small $\varepsilon > 0$, there exists a unique solution $y(t, \varepsilon)$ of problem (1)–(2). The asymptotic formula holds*

$$y(t, \varepsilon) = \sum_{i=1}^{N} \varepsilon^{i/k} \left[y_i + v_i\left(t\varepsilon^{-1/k}\right) + w_i(t\varepsilon^{-1})\right] + O\left(\varepsilon^{(N+1)/k}\right).$$

Proof. We seek an approximate solution of equation (1) in the form

$$\bar{y}_N = \sum_{i=1}^{N} \varepsilon^{i/k} y_i.$$

We have

$$By_1 = 0,$$
$$By_2 = F_{10} + F_{0k} y_1^k. \tag{13}$$

Hence $y_1 = c_1\varphi$. The solvability condition for the second equation of system (12) gives $\delta + \alpha_k c_1^k = 0$. Hence, for odd k we find one value, and for even k two values of c_1. From the second equation of system (13), $y_2 = \Gamma(F_{10} + F_{0k} y_1^k) + c_2\varphi$, c_2 is determined from the solvability condition for the third equation of system (13), and so on. For even k we have obtained two branches $\bar{y}_N$. From them one must choose one–the stable one. The case under consideration is characterized

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by the fact that the boundary-layer functions have different order in E^1 and in $E^{\infty-1}$.

Boundary layer in E^1. Put in (1) $t = \varepsilon^{1/k}\theta$, $y = \bar{y}_N + \bar{v}_N(\theta)$, where $\bar{v}_N = \sum_{i=1}^{N} v_i(\theta)\varepsilon^{i/k}$. We have

$$
\begin{gathered}
Bv_1 = 0, \\
Bv_2 = F_{10} + F_{0k}(v_1 + c_1\varphi)^k - \frac{dv_1}{d\theta}, \qquad (14) \\
\cdots\cdots\cdots\cdots\cdots\cdots
\end{gathered}
$$

From the first equation of system (14), $v_1 = e_1(\theta)\varphi$. The solvability condition for the second equation of system (14) gives

$$\frac{de_1}{d\theta} = \alpha_k(c_1 + e_1)^k + \delta, \qquad e_1(0) = -c_1. \qquad (15)$$

If k is odd, then for $\alpha_k < 0$ problem (15) has a unique solution for $t \geqslant 0$. In the case when k is even, for $\alpha_k\delta < 0$ and $c_1 = \operatorname{sign}\alpha_k \sqrt[k]{-\delta/\alpha_k}$ (the requirement of regular degeneracy) problem (15) also has a unique solution for $t \geqslant 0$. By the choice of the sign of c_1 we have chosen the branch $\bar{y}_N$. In this case the estimate holds

$$|e_1| \leqslant |c_1|e^{-h\theta}, \qquad h = |\alpha_k|\,|c_1|^{k-1}. \qquad (16)$$

The $v_i(\theta)$ are found similarly; however, for determining the functions $e_i(\theta)$ one obtains linear problems whose solutions are written explicitly in terms of $e_1(\theta)$.

Boundary layer in $E^{\infty-1}$. To remove the discrepancy in satisfying the boundary condition (2) by the function $\bar{y}_N + \bar{v}_N(\theta, \varepsilon)$, put in (1)

$$t = \varepsilon\tau, \qquad y = \bar{y}_N + \bar{v}_N(\theta, \varepsilon) + \bar{w}_N(\tau, \varepsilon),$$

where $\bar{w}_N = \sum_{i=1}^{N} w_i(\tau)\varepsilon^{i/k}$, $w_i(0) = -y_i - v_i(0)$. To determine $w_i(\tau)$ we obtain a system of the form (6), whence we successively find $w_i(\tau)$, with $w_1 \equiv 0$. The proof of existence and uniqueness of the solution, as well as the derivation of the error estimate, are carried out according to the scheme of Theorem 2.

Equation (1) contains, as a special case, an autonomous system of ordinary differential equations, integro-differential equations, and also equations of parabolic type, for which it is not difficult to reformulate the results obtained. To the problem considered there reduces[5] the corresponding problem on perturbation of a linear equation by a small nonlinear term containing the derivative with respect to t.

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Note: Figure translations are in progress. See original paper for figures.

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