V. Ya. VOLK

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Abstract

Full Text

V. Ya. VOLK

ON THE SPECTRAL DECOMPOSITION FOR A CLASS OF NON-SELF-ADJOINT OPERATORS

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1. In this paper families of invariant subspaces and projection operators are constructed for certain classes of linear non-self-adjoint operators.

Let T be a bounded operator acting in a reflexive Banach space H. By the spectrum of the operator T we shall mean the set of singular points of its resolvent $R(\lambda) = (T - \lambda E)^{-1}$, and by the spectrum of the operator T^* the set of singular points of $R^*(\lambda) = (T^* - \bar{\lambda}E)^{-1}$.

We shall assume that the spectrum of the operator T is zero-dimensional. This means that, whatever the domain Δ in the complex plane may be, the part of the spectrum of the operator T lying in Δ can be separated from the remaining spectrum by a rectifiable contour on which there lies only a finite number of points of the spectrum of the operator T. We shall not distinguish between two domains Δ_1 and Δ_2 if they contain one and the same part of the spectrum of the operator T.

The method for constructing invariant subspaces is as follows. Let Δ be a domain of the complex plane with rectifiable boundary, and let $\varphi_\Delta(\lambda)$ be a function analytic and non-vanishing in the domain Δ, continuous in the closed domain $\bar{\Delta}$, and such that the product $|\varphi_\Delta(\lambda)| \cdot \|R(\lambda)\|$ is bounded on the boundary of the domain. Then the operator

$$\Phi_\Delta = \int_{C_\Delta} \varphi_\Delta(\mu) R(\mu)\, d\mu$$

(C_Δ is the boundary of the domain Δ) is bounded. The closure of its range H_Δ is an invariant subspace of the operator T. If the functional h^* vanishes on all of H_Δ, then the function $R^*(\lambda)h^*$ is regular in Δ. Indeed, if $\langle h, h^* \rangle$ denotes the value of the functional h^* on the vector h, then, obviously,

$$\int_{C_\Delta} \langle \varphi_\Delta(\mu) R(\mu) h,\ R^*(\lambda) h^* \rangle\, d\mu = 0$$

or

$$\int_{C_\Delta} \langle \varphi_\Delta(\mu) R(\lambda) R(\mu) h,\ h^* \rangle \, d\mu = 0.$$

Applying the resolvent equation

$$R(\mu)R(\lambda) = \frac{R(\mu) - R(\lambda)}{\mu - \lambda},$$

we obtain

$$\int_{C_\Delta} \frac{\langle \varphi_\Delta(\mu) R(\lambda) h,\ h^* \rangle}{\mu - \lambda} \, d\mu = \int_{C_\Delta} \frac{\langle \varphi_\Delta(\mu) R(\mu) h,\ h^* \rangle}{\mu - \lambda} \, d\mu.$$

Computing the integral on the left, we obtain the equality

$$\langle \varphi_\Delta(\lambda) R(\lambda) h,\ h^* \rangle = \frac{1}{2\pi i} \int_{C_\Delta} \frac{\langle \varphi_\Delta(\mu) R(\mu) h,\ h^* \rangle}{\mu - \lambda} \, d\mu. \tag{1}$$

On the right-hand side of the equality is a Cauchy-type integral, defining a function regular in Δ. Noting that equality (1) is valid for any $h \in H$ and that

$$\langle \varphi_\Delta(\lambda) R(\lambda) h, h^* \rangle = \varphi_\Delta(\lambda) \langle h, R^*(\lambda) h^* \rangle,$$

we conclude that $R^*(\lambda)h^*$ is regular in Δ.

If the entire spectrum of the operator T is covered by the system of domains $\Delta_1, \Delta_2, \dots, \Delta_n, \dots$, and for each domain Δ_k one can construct, by the method described above, an invariant subspace H_{Δ_k}, then the linear span of all H_{Δ_k} is dense in H. Indeed, if h^* annihilates all H_{Δ_k}, then $R^*(\lambda)h^*$ is regular on the whole spectrum and, consequently, on the entire complex plane; and then, as is known, $h^* = 0$.

It is also easy to show that if Δ_1 and Δ_2 are two domains and Δ_0 is their intersection, then H_{Δ_0} is contained in the intersection of H_{Δ_1} and H_{Δ_2}.

We shall say that the operator T is an S-operator if for any domain Δ one can construct a function $\varphi_\Delta(\lambda)$ possessing the properties indicated above. For the construction of the functions $\varphi_\Delta(\lambda)$, it proves necessary to relate the growth of the resolvent of the operator near the spectrum to the location of the spectrum in the complex plane.

Let the point ξ lie on any ray K emanating from a point of the spectrum λ_0 and encountering no further points of the spectrum on its way. We shall say that: a) the resolvent has **polynomial order of growth near the spectrum** if on K the inequality

$$\|R(\lambda_0+\xi)\| \leqslant \frac{C}{|\xi|^p}, \qquad p>0,\ C>0;$$

is satisfied; b) the resolvent has **exponential order of growth** α, if on K the inequality

$$\|R(\lambda_0+\xi)\| \leqslant Ce^{C_1|\xi|^{-\alpha}}, \qquad C>0,\ C_1>0,\ \alpha>0;$$

is satisfied; c) the resolvent **satisfies Levinson’ s condition** if there exists an improper integral

$$\int_{K_1} |\ln\ln\|R(\lambda_0+\xi)\|| \cdot |d\xi|,$$

where K_1 is a segment of the ray K, one of whose endpoints is the point λ_0.

We shall say that a set X in the complex plane **satisfies the Lipschitz condition with coefficient** β, if for any point $\xi_0 \in X$ there exists a neighborhood of it in which the values of the function $\arg(\xi-\xi_0)$, for $\xi \in X$, are contained within a strip of width β.

Theorem 1. *If the spectrum of the operator T is zero-dimensional and its resolvent near the spectrum grows polynomially, then T is an S-operator.*

Theorem 2. *If the spectrum of the operator T satisfies the Lipschitz condition with coefficient β and its resolvent near the spectrum has exponential order of growth α, then for $\alpha<\pi/\beta$, T is an S-operator.*

Theorem 3*. *If the spectrum of the operator T lies on a smooth curve and the growth of its resolvent near the spectrum satisfies Levinson’ s condition, then T is an S-operator.*

In the case of Theorem 1, $\varphi_\Delta(\lambda)$ is a polynomial; in the case of Theorem 2, $\varphi_\Delta(\lambda)$ is expressed in terms of exponentials; and in the case of Theorem 3, in terms of functions that Levinson constructed in (^{4}).

2. The presence of a system of invariant subspaces for the operator T does not yet ensure the existence of projection operators commuting with T. For example, the operator of multiplication by the independent variable in

in the space $C(0,1)$ is an S-operator, but it has no projection operators except the trivial ones: 0 and E.

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* Theorem 3 was previously published by Yu. I. Lyubich and V. I. Matsaev in (^{3}).

To construct projection operators we shall require that the set of eigenvalues of the operators T and T^* be at most countable and that the continuous spectra of the operators T and T^* coincide. Let the operator T satisfy the conditions of Theorem 1, and let Δ be a domain on whose boundary there lie only points of the continuous spectrum, in finite number. We construct the domain Δ' so that the domain $\Delta \cup \Delta'$ contains the entire spectrum of the operator T, except for those of its points which lie on the boundary of the domain Δ, and so that the intersection of the domains Δ and Δ' is empty. Construct the invariant subspaces H_Δ and $H_{\Delta'}$. Their linear hull is dense in H. Indeed, if the functional h^* is annihilated on H_Δ and $H_{\Delta'}$, then $R^*(\lambda)h^*$ is regular in Δ and Δ'.

From equality (1) it follows that points of the spectrum lying on the boundary of Δ cannot be essential singularities of $R^*(\lambda)h^*$ ($\varphi_\Delta(\lambda)$ is a polynomial). Nor can they be poles, since they belong to the continuous spectrum. Consequently, $R^*(\lambda)h^*$ is regular in the entire complex plane, and $h^* = 0$.

It is just as easy to show that the intersection of H_Δ and $H_{\Delta'}$ is empty. Consequently, one can construct a closed (but not necessarily bounded) projection operator $E(\Delta)$ such that $E(\Delta)H = H_\Delta,\ E(\Delta)H_{\Delta'} = 0$.

Thus, the following is valid.

Theorem 4. *Let the operator T satisfy the conditions of Theorem 1. If, in addition, the operators T and T^* have at most a countable number of eigenvalues, and the continuous spectra of the operators T and T^* coincide, then for the operator T one can construct a family of closed projection operators such that to every domain Δ there corresponds an operator $E(\Delta)$.*

If the operator $E(\Delta)$ can be extended to all of H with closedness preserved, then $E(\Delta)$ is bounded. With the aid of Theorem 4 the following can be proved.

Theorem 5. *Let the spectrum of the closed operator T be situated on a smooth curve Γ, and let its resolvent be representable in the form*

$$R(\lambda) \int \frac{d\sigma(\xi)}{\xi - \lambda}, \tag{2}$$

where $\langle \sigma(\xi)h, h^* \rangle$ is a function of bounded variation for any $h \in H,\ h^* \in H^*$. Then for the operator T one can construct a uniformly bounded family of projection operators such that to every measurable set δ there corresponds an operator $E(\delta)$.

As an example, consider the operator

$$T = -y'' + q(x)y$$

in $L_2(0, \infty)$ with boundary condition $y'(0) - \theta y(0) = 0$ and $q(x)$ such that

$$\int_0^\infty e^{\varepsilon x}|q(x)|\,dx < \infty$$

for some $\varepsilon > 0$. This operator was studied by M. A. Naimark (^{1}).

Using the asymptotics of the solutions obtained in (^{1}), it is easy to show that the resolvent of the operator T can be represented in the form (2), if one disregards a finite number of eigenvalues. Using Theorem 5 and the method of work (^{2}), one can show that the system of eigenfunctions in the sense of (^{2}) and associated functions of the operator T is complete in $L_2(0,\infty)$.

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Note: Figure translations are in progress. See original paper for figures.

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