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Abstract

Full Text

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ON THE INDICES OF FUNCTIONS ON RIEMANN SURFACES

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1. For a single-valued analytic function $F(z) = f'(z)$, defined in a multiply connected domain D with boundary L, we write the argument principle as

$$\frac{1}{2\pi i}\int_L d\ln f'(z) = \sum_{k=1}^{n} N(a_k). \tag{1}$$

Here $N(a_k)$ is the order of a zero or the order of a pole (with a minus sign) of the function $f'(z)$ in the domain D.

The left-hand side of formula (1) determines the rotation along L of the vector field $v(z) = f'(z)$, whose complex potential is $f(z)$:

$$\gamma = -\frac{1}{2\pi i}\int_L d\ln f'(z). \tag{2}$$

Then (1) can be rewritten in the form

$$\gamma = \sum_{k=1}^{n} \gamma(a_k), \tag{3}$$

where $\gamma(a_k)$ is the index of the point a_k,

$$\gamma(a_k) = -N(a_k) = -\frac{1}{2\pi i}\int_{c_k} d\ln f'(z), \tag{4}$$

where c_k is a circle of sufficiently small radius with center at a_k.

M. Morse ([1]) extended equality (3) to interior mappings. In a number of papers ([2–8]) relation (3) is refined and generalized.

In the present note an analogous equality is obtained for automorphic functions and functions on Riemann surfaces, and applications are indicated.

Remark. In solving boundary-value problems, indices are also used ([9], p. 98). By definition, the index χ of a function $G(z)$ is the rotation of the vector field that is generated by the vectors $v(z) = G(z)$.

2. Let a simple automorphic function $z(t)$ be given in a multiply connected domain $\dot{D}$. Form the vector field $z'(t)$. In the fundamental polygon of the group of fractional-linear transformations (to which the automorphic function belongs), the domain D may also be disconnected. We divide it into simply connected parts by cuts lying entirely in the fundamental polygon. First discard circular neighborhoods of the points that are singularities of the function $\ln z'(t)$. Consider $\int d \ln z'(t)$ along the boundaries of these simply connected domains. Taking the sum of the integrals and taking (2) and (4) into account, we obtain

$$\gamma = \sum_{k=1}^{n} \gamma(t_k) + \alpha(D), \tag{5}$$

where γ is the rotation of the vector field along the entire boundary of the domain D; $a(D)$ is a constant depending on the domain D. This constant is obtained by adding the integrals $\int d \ln z'(t)$ over congruent arcs of the boundary of the fundamental polygon. If the total angular measure of those parts of the congruent arcs on which the function $z(t)$ is defined is denoted by α, then (([10]), p. 124)

$$a(D) = \frac{\alpha}{2\pi}.$$

At the point t_k the function $z'(t)$ will have the representation

$$z'(t) = (t - t_k)^{p_k - 1} z_k(t), \qquad z_k(t_k) \neq 0,$$

and therefore the index at this point, on the basis of (4), is equal to

$$\gamma(t_k) = 1 - p_k. \tag{6}$$

3. Let two domains on finite-sheeted Riemann surfaces correspond to one another by means of the function $w = f(z)$. Uniformize this function by two automorphic functions $z = z(t)$ and $w = w(t)$, defined in a multiply connected domain D_t with boundary L_t. For the uniformizing functions we have two relations of the form (5)

$$\gamma_z = \sum_{k=1}^{n} \gamma_z(t_k) + a(D), \qquad \gamma_w = \sum_{k=1}^{n} \gamma_w(t_k) + a(D), \tag{7}$$

where the subscript indicates for which function the relation is written.

Define the rotation γ for the field $\overline{f'(z)}$ by formula (2), with L being the boundary of the domain on the Riemann surface. Then

$$\gamma = \gamma_w - \gamma_z. \tag{8}$$

Subtracting one equality in (7) from the other, we obtain

$$\gamma = \sum_{k=1}^{n} \gamma(z_k), \tag{9}$$

where $z_k = z(t_k)$. At the point z_k the function $f(z)$ will have the representation

$$f(z) - w_k = (z - z_k)^{p_k/q_k} F_k\left[(z - z_k)^{1/q_k}\right], \tag{10}$$

where $q_k \geqslant 1$, $|p_k|$ and q_k are relatively prime integers, and $F_k(0) \neq 0$. The index at this point, in accordance with formulas (6) and (8), is computed as

$$\gamma(z_k) = q_k - p_k.$$

It is necessary to take into account that

$$z_k = z_{k+1} = \ldots = z_{k+l_k-1},$$

because over each point z_k there are l_k points with representations of the form (10). If m is the number of sheets of the Riemann surface over the z-plane, then $1 \leqslant l_k \leqslant m$.

Relation (9) can be proved directly on the Riemann surface by using the vector field $\overline{f'(z)}$.

Instead of the representation (10), it is convenient to use the representation

$$f'(z) = (z - z_k)^{-\gamma(z_k)/q_k} \Phi_k\left[(z - z_k)^{1/q_k}\right], \qquad \Phi_k(0) \neq 0. \tag{11}$$

Incidentally, such a representation permits the presence of logarithmic singularities in $f(z)$.

All the relations obtained in ([7],[8]) are derived from (9).

4. For closed Riemann surfaces over the planes z and w, with a finite number of branch points, relation (9) is written in the form

$$\sum_{k=1}^{n} \gamma(z_k) = 0, \tag{12}$$

where among the z_k there is ∞. Hence, using representation (11) and the formula for the connectivity order of a Riemann surface (([10]), $p.243$), we obtain for the covariant $f'(z)$ the well-known formula

$$N' - P' = 2p - 2,$$

where N' is the sum of the orders of the zeros of the covariant, P' is the sum of the orders of the poles, and p is the genus of the Riemann surface.

Applying (12) to a meromorphic function on a closed Riemann surface, we obtain that the sum of the orders of its zeros is equal to the sum of the orders of its poles.

5. The application of the relation $\gamma = \sum_{k=1}^{n} \gamma(z_k)$ is connected with the calculation

$$\gamma = n_z - n_w, \tag{13}$$

where $2\pi n_z$ and $2\pi n_w$ represent the changes of the angles made by the tangents to L_z and L_w with the positive direction of the real axis. L_z is the boundary of a domain on the Riemann surface, and L_w is the image of L_z under the mapping $w = f(z)$. Formula (13) follows from (2) and the equality

$$2\pi n_w = 2\pi n_z + \text{var}_{L_z} \arg f'(z).$$

Let us make some remarks concerning the definition of n_w. M. Morse ([11]), $p.55$, calls the quantity $(-n_w)$ the boundary index. If the curve L_w is easy to draw, then the number of rotations of the tangent under a complete circuit of L_w will be equal to n_w. If the curve L_w is difficult to draw, then we use the formula

$$n_w = m - M = M_1 - m_1, \tag{14}$$

where m (m_1) is the number of incoming (outgoing) minima, and M (M_1) is the number of incoming (outgoing) maxima of the function $u = \text{Re}\, f(z)$ on the contour L_z, the preimage of L_w. An extremum is called incoming (outgoing) if for it $\partial u/\partial n > 0$ $(\partial u/\partial n < 0)$, where n is the inward normal to L_z. It is assumed that at every extremal point $|f'(z)| = |\partial u/\partial n| \neq 0$. The first part of formula (14) was substantiated by M. Morse (([1]), $pp.55 and 77$). However, it is easy to prove it geometrically.

Calculations by formula (14) are carried out using only the boundary values of $f(z)$ on L_z ([2]). For this, one of the Cauchy-Riemann conditions is applied. Namely, $\partial u/\partial n = -\partial v/\partial s$, where s is the arc of the contour L_z.

Let us indicate applications.

1°. Equality (9) is used to solve the question of the admissible singularities of the desired function in inverse boundary-value problems. By singularities one understands all singular points of the function $\ln z'(w)$.

We shall assume that the function $w = u(s) + iv(s)$ determines a closed contour in the w-plane. Along this contour we find n_w (geometrically or analytically, as was already noted). n_z is prescribed additionally.

Singularities are placed in the unknown domain D_z, and the type of the singularities and their number must satisfy the relation

$$\sum_{k=1}^{n} \gamma(z_k) = n_z - n_w.$$

Not all combinations of singularities can be realized. The arrangement of the singularities will not, in general, be unique. Nevertheless, in the simply connected case the "model" of the domain is determined only by n_z. For a given n_z, the model of the sought domain will be an n_z-sheeted circle with a branch point of order $(n_z - 1)$ at the center. However, one cannot assert that the domain obtained will be exactly n_z-sheeted. For this, the use of more delicate methods is required. In particular, for $n_z = 1$ it is necessary to determine additionally when the domain D_z will be one-sheeted ([11]).

In the multiply connected case, n_z is understood to mean the sum

$$n_z = n_1 - n_2 - \dots - n_\nu,$$

where n_1 may be positive or negative, $1 \le n_k \le |n_1|$, $k = 2, \dots, \nu$. The model will be an n_1-sheeted circle with $(\nu - 1)$ circular cuts in the form of n_k-sheeted circles, $k = 2, \dots, \nu$.

2°. Let us determine the number of zeros N of a finit-valued function $f(z)$ inside a certain domain D, in which $f(z)$ may have poles and branch points. Construct the Riemann surface of the function

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

over the domain D, without taking into account the logarithmic singularities of the function $F(z)$. Then

$$N = \gamma - \sum_{k=1}^{l} \gamma(z_k),$$

where z_k $(k = 1, \dots, l)$ are algebraic branch points, poles, and logarithmic singularities of $F(z)$, and γ is the rotation of the vector field $\overline{f(z)}$ along the boundary of the Riemann surface over the domain D.

To determine the number of zeros of the derivative of an automorphic function in a certain domain, one may use equality (5).

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