MATHEMATICS

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1963

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Abstract

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ON EIGENVALUE PROBLEMS FOR SECOND-ORDER EQUATIONS IN THE CASE OF NONLINEAR DEPENDENCE ON THE PARAMETER λ

(Presented by Academician I. G. Petrovskii, March 15, 1963)

1. Let us consider the boundary-value eigenvalue problem for a second-order differential equation in the case when the parameter λ enters the equation nonlinearly:

$$\frac{d^2w}{dx^2} + k^2 Q(x,\lambda) w = 0, \qquad w(-\infty) = w(\infty) = 0, \tag{1}$$

and we shall seek its eigenvalues in a certain domain G in the plane of the complex variable λ, for a fixed real value of the parameter k. The aim of the present work is to develop an asymptotic method for the approximate determination of the eigenvalues of such a problem for large values of k. In the case of the classical Sturm–Liouville problem with a discrete spectrum, the equation approximately determining the eigenvalues for large k has the form of the Bohr quantization conditions ([1,2]). It is of interest to derive analogous equations for the approximate determination of eigenvalues for non-self-adjoint problems, when the function $Q(x,\lambda)$ is, generally speaking, complex.

We investigate problem (1) under the following assumptions on the function $Q(x,\lambda)$:

1) There exist such x_1 and x_2 that, for any $\lambda \in G$, the function

$$\operatorname{Im} \int_{x_0}^{x} \sqrt{Q(t,\lambda)}\, dt$$

is a monotone function of x for $-\infty < x < x_1$ and $x_2 < x < \infty$, and moreover

$$\lim_{x\to\pm\infty} \left[\left| \operatorname{Im} k \int_{x_0}^{x} \sqrt{Q(t,\lambda)}\, dt \right| \mp \frac{1}{4} \ln |Q(x,\lambda)| \right] = +\infty.$$

2) The integrals

$$\int_{-\infty}^{x_1} {Q'}^2(t,\lambda)\, Q^{-5/2}(t,\lambda)\, dt, \qquad \int_{-\infty}^{x_1} Q''(t,\lambda)\, Q^{-3/2}(t,\lambda)\, dt,$$

$$\int_{x_2}^{\infty} {Q'}^2(t,\lambda)\, Q^{-5/2}(t,\lambda)\, dt, \qquad \int_{x_2}^{\infty} Q''(t,\lambda)\, Q^{-3/2}(t,\lambda)\, dt$$

converge absolutely and uniformly in the domain G.

3) The analytic continuation of the function $Q(x,\lambda)$ to the plane of the complex variable $z = x + iy$, $Q(z,\lambda)$, is an entire function of z for $\lambda \in G$.

4) The function $Q(z,\lambda)$ is an analytic function of λ in the domain G for every z.

5) The function $Q(z,\lambda)$ has only simple zeros $z_i(\lambda)$, depending analytically on λ in the domain G.

We note that, by virtue of requirements 1) and 2), each of the boundary conditions

problem (1) singles out a solution of equation (1), unique up to a factor. We shall denote these solutions by w_1 and w_2.

2. To investigate problem (1), we consider equation (1) on the entire plane of the complex variable and construct asymptotic expansions of the solutions w_1 and w_2 in negative powers of k. In deriving the required asymptotic formulas, an important role is played by the topological structure of the level lines of the harmonic function

$$\operatorname{Im} \int_{z_0}^{z} \sqrt{Q(t,\lambda)}\, dt,$$

which is determined by the location of the zeros $z_i(\lambda)$ of the function $Q(z,\lambda)$.

Fix some value λ and consider, for this value of λ, in the plane of the complex variable z, the system of lines $\Gamma_i(\lambda)$ defined by the equations

$$\operatorname{Im} \int_{z_i(\lambda)}^{z} \sqrt{Q(t,\lambda)}\, dt = 0 \quad (i = 1, 2, \ldots).$$

The connected branch of this line containing the point $z_i(\lambda)$ will be denoted by $\Gamma_i(\lambda)$.

Suppose that the line $\Gamma_{i_0}(\lambda)$, corresponding to the zero $z_{i_0}(\lambda)$, does not pass through other zeros of the function $Q(z,\lambda)$; then the zero $z_{i_0}(\lambda)$ will be called an **isolated zero** of the function $Q(z,\lambda)$ for the given value of λ. The line $\Gamma_{i_0}(\lambda)$, corresponding to the isolated zero $z_{i_0}(\lambda)$, consists of three branches which emanate from the point $z_{i_0}(\lambda)$ and go to infinity without self-intersections or mutual intersections.

Assume now that the line $\Gamma_{i_1}(\lambda)$, in addition to the zero $z_{i_1}(\lambda)$, also passes through the zeros $z_{i_2}(\lambda), \dots, z_{i_s}(\lambda)$, and does not pass through the remaining zeros of the function $Q(z, \lambda)$. We shall then say that the zeros $z_{i_1}(\lambda), \dots, z_{i_s}(\lambda)$ of the function $Q(z, \lambda)$ **form, for the given value of** λ, an **s-point complex**. The lines $\Gamma_{i_2}(\lambda), \dots, \Gamma_{i_s}(\lambda)$, corresponding to the zeros $z_{i_2}(\lambda), \dots, z_{i_s}(\lambda)$, coincide with the line $\Gamma_{i_1}(\lambda)$, and therefore we shall denote this line by $\Gamma_{i_1 \dots i_s}(\lambda)$.

The line $\Gamma_{i_1 \dots i_s}(\lambda)$ has the following structure: from each zero $z_{i_\alpha}(\lambda)$ $(1 \le \alpha \le s)$ three branches of the line emanate, which either end at other zeros of the complex or go to infinity; moreover, according to condition 1, none of these branches can go to infinity either straight along the real axis or while oscillating infinitely about it.

The line $\Gamma_{i_1 \dots i_s}(\lambda)$ divides the whole plane of the complex variable z into $s+2$ regions. The union of those regions of the partition which contain arbitrarily large, in modulus, real numbers (there are either one or two such regions of the partition) will be called the **outer region** of the given complex. In those cases when the outer region consists of two adjacent regions of the partition, the points of the branch of the line $\Gamma_{i_1 \dots i_s}(\lambda)$ separating them will also be assigned to the outer region of the complex. We shall call the complex a **complex of the first or second kind**, according as its outer region is, or is not, connected.

3. **Theorem 1.** *If, for no value λ belonging to a connected closed subdomain g of the domain G, the function $Q(z, \lambda)$ has complexes of the second kind, then there exists a $k_0 = k_0(g)$ such that for every $k > k_0(g)$ problem (1) has no eigenvalues in the domain g.*

Theorem 2. *Let a connected closed subdomain g of the domain G satisfy the following requirements:*

1) *For every value $\lambda \in g$, the roots $z_1(\lambda)$ and $z_2(\lambda)$ of the function $Q(z, \lambda)$ are either isolated roots or form between themselves a two-point complex of the second kind.*
2) *The remaining roots of the function $Q(z, \lambda)$ do not form complexes of the second kind for any value of λ.*

Then the equation determining the eigenvalues of problem (1) has, in the domain g, the form

$$D(\lambda, k) = \cos k S_{1,2}(\lambda) + \frac{1}{k} \alpha(\lambda, k) e^{ikS_{1,2}(\lambda)} + \beta(\lambda, k) e^{-ikS_{1,2}(\lambda)} = 0. \qquad (2)$$

Here

$$S_{1,2}(\lambda) = \int_{z_1(\lambda)}^{z_2(\lambda)} \sqrt{Q(t,\lambda)}\, dt,$$

where the path of integration must not intersect any of the lines $\Gamma_i(\lambda)$ ($i = 3, 4, \ldots$); $\alpha(\lambda, k)$ and $\beta(\lambda, k)$ are functions analytic and bounded with respect to their arguments for $\lambda \in g$, $k > k_0(g)$.

Let us compare with equation (2) the equation

$$D_0(\lambda, k) = \cos kS_{1,2}(\lambda) = 0. \tag{3}$$

If the derivative of the function $S_{1,2}(\lambda)$ does not vanish in the domain g, then equation (3) determines in this domain the roots of equation (2) with accuracy up to terms not exceeding c/k^2, where $c = c(g)$ is a constant.

The results obtained are easily generalized to the case when several pairs of zeros of the function $Q(z, \lambda)$: $z_1(\lambda)$ and $z_2(\lambda)$, $z_3(\lambda)$ and $z_4(\lambda)$, ..., can form two-point complexes of the second kind in the domain g. In this case the approximate equation determining the eigenvalues of problem (1), for large k, will have the form

$$D_0(\lambda, k) = \cos kS_{1,2}(\lambda) \cos kS_{3,4}(\lambda) \cdots = 0.$$

Thus, equations of type (3) make it possible to determine approximately the eigenvalues of problem (1) everywhere, except in neighborhoods of those values λ for which the function $Q(z, \lambda)$ has multi-point complexes of the second kind. To determine the eigenvalues in a neighborhood of such λ, the corresponding approximate equations are easily derived. The form of these equations depends essentially not only on the number of zeros of the function $Q(z, \lambda)$ forming the complex, but also on their mutual arrangement, i.e., on the structure of the complex.

In conclusion we note the following. Equation (3) formally has the form of the Bohr quantization conditions, i.e., the eigenvalues of self-adjoint and non-self-adjoint problems are determined by approximate equations of one and the same type. However, if in the first case it is immediately clear for which pairs of zeros of the function $Q(z, \lambda)$ one should write equations of type (3), then in the second case, when compiling these equations, it is necessary to take into account the character of the complexes. If some pair of zeros of the function $Q(z, \lambda)$ forms a complex of the second kind, then the roots of equation (3) approximately determine the eigenvalues of problem (1); if, however, this pair forms a complex of the first kind, then the roots of equation (3) have no relation whatsoever to the eigenvalues of problem (1). Therefore the formal application of equation (3) to finding the eigenvalues of problem (1), without analyzing the character of the complexes, can lead to erroneous results.

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Received
5 III 1963

References Cited

1. H. A. Kramers, Zs. Phys., **39**, 518 (1926).
2. V. P. Maslov, DAN, **94**, 623 (1954).

Note: Figure translations are in progress. See original paper for figures.