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Abstract

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On the Strong Degeneracy of Simple p-Lie Algebras

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1. A simple p-Lie algebra Ω is called degenerate if the bilinear trace form $\operatorname{Tr} XY$ defined on it is identically equal to zero. Degeneracy is a property inherent in a broad class of simple Lie algebras defined over fields of finite characteristic. We shall also say that a simple p-Lie algebra Ω has strong degeneracy if there exists in it an element $c \neq 0$ such that $C^2 = 0$ (an algebra with condition (*) in paper (^{6})). Here and below the letter X denotes the adjoint endomorphism of an element $x \in \Omega$. The ground field Φ is assumed to be algebraically closed of characteristic $p > 5$. We note that the restriction $p > 3$ enters implicitly into the definition of strong degeneracy, whereas the inequality $p > 5$ is caused only by the desire to avoid inessential details. As is not difficult to see, and as was already noted in (^{6}), ordinary degeneracy follows from strong degeneracy. The converse assertion, generally speaking, is false; however, there are serious grounds to expect that the indicated classes of algebras are close to one another. The purpose of the present note is to obtain the first facts in this direction. Some of the results remain true for any p-Lie algebra with zero center. For convenience in formulation we introduce the following definition. Let $\mathfrak{H}$ be a Cartan subalgebra of the Lie algebra Ω,

$$\Omega = \mathfrak{H} + \sum_{\alpha \neq 0} \Omega_\alpha$$

the corresponding Cartan decomposition. We shall call an element $x \in \Omega$ (a subspace $\mathfrak{M} \subset \Omega$) $\mathfrak{H}$-homogeneous if x belongs either to $\mathfrak{H}$, or to some invariant subspace Ω_α (respectively, if

$$\mathfrak{M} = \mathfrak{M} \cap \mathfrak{H} + \sum_{\alpha \neq 0} \mathfrak{M} \cap \Omega_\alpha$$

).

Theorem 1. *Suppose that in a simple p-Lie algebra Ω the following conditions are fulfilled:*

1) *there exists in* Ω *a Cartan subalgebra* $\mathfrak{H}$ *satisfying Engel' s* $(p-1)$*-st condition, i.e.* $h_1 H_2^{p-1} = 0$ *for all elements* h_1, h_2 *of* $\mathfrak{H}$*;*
2) $A^{p-1} = 0$ *for at least one* $\mathfrak{H}$*-homogeneous element* $a \neq 0$ *of* Ω*;*
3) $\operatorname{Tr} X^k = 0$ *for* $k = 1, 2, 3, 4$ *and all* $x \in \Omega$*.*

Then the algebra Ω *is strongly degenerate.*

Theorem 2. *A simple p-Lie algebra* Ω *has strong degeneracy if in it there exists an abelian Cartan subalgebra* $\mathfrak{H}$*, one of whose semisimple components is one-dimensional. More precisely, if* $\dim \mathfrak{H} > 1$*, then* $C^2 = 0$ *for some* $\mathfrak{H}$*-homogeneous element* $c \neq 0$*.*

2. Before outlining the proof of Theorems 1 and 2, we shall make several remarks concerning their formulations. Conditions 1) and 2) of the first theorem are quite natural. In all known simple Lie algebras the Cartan subalgebras are commutative, and the roots α are correspondingly linear functions on $\mathfrak{H}$. It is precisely this latter circumstance that is essential for our proof, and condition 1) guarantees the linearity of the functions α. In some degenerate simple p-Lie algebras one can indicate abelian Cartan subalgebras with respect to which condition 2) is not fulfilled (2). Replacing condition 3), whose origin is connected with Proposition 2 of paper (6), by the condition of ordinary degeneracy ($\operatorname{Tr} X^k = 0,\ k = 1, 2$)

leads at once to algebras without a strong expression. The problem of describing such algebras has not yet been solved.

Let us further note that the Cartan subalgebra $\mathfrak{H}$ of a p-algebra Lie $\mathfrak{L}$ is also a p-algebra. If, in addition, it is commutative, then there is a direct decomposition

$$\mathfrak{H} = \mathfrak{U} + \mathfrak{B},$$

where $\mathfrak{U}$ is the semisimple component with basis $u_1, \dots, u_n$ ($u_i^p = u_i$), and $\mathfrak{B}$ is the nilpotent component ($v^{p^k} = 0$ for all $v \in \mathfrak{B}$ and sufficiently large k) (see (3)). In the Cartan subalgebra, obviously, $n \geqslant 1$, and the assertion of Theorem 2 pertains to the case $n = 1$. The case $\dim \mathfrak{H} = 1$, corresponding to the Witt algebra (4), is excluded from consideration, since in what follows it is assumed that $\dim \mathfrak{B} > 0$. Such algebras do exist. For example, in one of the simple p-algebras constructed by R. Block (1),

$$\mathfrak{H} = \{u; v_1, \dots, v_{p-2}\}, \qquad u^p = u, \qquad v_i^p = 0.$$

3. The proof of Theorem 1 is based on the following assertion, which is of independent interest.

Lemma 1. *Let* $\mathfrak{L}$ *be a simple p-algebra Lie satisfying conditions 1) and 2) of Theorem 1, but not possessing a strong expression. Then the algebra* $\mathfrak{L}$*, considered as a vector space, admits a decomposition into a direct sum of* $\mathfrak{H}$*-homogeneous subspaces:*

$$\mathfrak{L} = \mathfrak{M} + \mathfrak{M}_1 + \mathfrak{M}_{-1} + \mathfrak{M}_0,$$

where

$$\mathfrak{M}_0 = \{a, b, h\}, \quad [ab] = h, \quad [ah] = 2a, \quad [bh] = -2b, \quad A^3 = 0, \quad B^3 = 0,$$

$$h^p = h; \quad \dim \mathfrak{M}_1 = \dim \mathfrak{M}_{-1}, \quad xH = x, \quad yH = -y \quad \text{for all } x \in \mathfrak{M}_1,\ y \in \mathfrak{M}_{-1};$$

$$\mathfrak{M}_{-1}A = \mathfrak{M}_1, \qquad \mathfrak{M}_1 B = \mathfrak{M}_{-1}, \qquad \mathfrak{M}_1 A = 0, \qquad \mathfrak{M}_{-1}B = 0; \qquad \mathfrak{M}_0 A = \mathfrak{M}_0 B = 0;$$

each element $z \in \mathfrak{M}_0$ *is a linear combination of elements of the form*

$$b\,(X_1X_2 + X_2X_1), \qquad x_i \in \mathfrak{M}_1.$$

The proof of Lemma 1 is obtained from the following considerations. Using condition 2) and the first half of the proof of Theorem 2 from the work (5), in the algebra $\mathfrak{L}$ one can construct an $\mathfrak{H}$-homogeneous element of nilpotency index three. From the whole set of such elements one is chosen (denote it by b; $B^3 = 0$) for which the dimension of the space $\mathfrak{L}B^2$ is minimal. The indicated choice ensures the existence of an element a, normalized in such a way that $aB^2 = 2b$. The subspaces $\mathfrak{M}_i$ of the algebra $\mathfrak{L}$ that are mentioned in Lemma 1 are completely characterized by the action on them of the operator B. Condition 1) guarantees us the equality $\dim \mathfrak{L}_a B^2 = 1$ whenever $\mathfrak{L}_a B^2 \neq 0$. Hence it follows without difficulty that $\dim \mathfrak{L}B^2 = p^\nu$, $\nu \geq 0$. Such a special form of the dimension of the space $\mathfrak{L}B^2$ suggests the possibility of constructing, in the case $\nu > 0$, an element with nilpotency index two. This is indeed so, although, unfortunately, in the proof one has to go beyond the domain of $\mathfrak{H}$-homogeneous elements. Thus, by assumption, the algebra does not possess a strong expression, so the only remaining possibility is $\nu = 0$. This is the decisive point in the proof of the lemma. The properties of the attached endomorphisms A, B and H ($h = aB$), recorded in its formulation, are obtained without difficulty. The assertion about the subspace $\mathfrak{M}_0$ is a consequence of the simplicity of the algebra $\mathfrak{L}$.

In fact, Lemma 1 contains much more information than is needed for the proof of Theorem 1, and the further use of the simplicity of the algebra $\mathfrak{L}$ should lead to more detailed information about the subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_0$.

With the aid of Lemma 1 it is easy to compute that

$$\operatorname{Tr} H^2 = 8 + 2\dim \mathfrak{M}_1, \qquad \operatorname{Tr} H^4 = 32 + 2\dim \mathfrak{M}_1.$$

Therefore assertion of Theorem 1 follows immediately from condition 3).

Suppose now that Theorem 2 is false, but that $\overline{A^{p-1}} = 0$ at least for one $\mathfrak{H}$-homogeneous element $a \neq 0$. Then in $\mathfrak{L}$ one can distinguish the subspaces indicated in Lemma 1. The difference consists only in the fact that the equality $\nu = 0$, for whose proof in the general case (as was noted above) one has to involve nonhomogeneous elements, is established directly in the present situation. For this it is only necessary to observe that

$\mathfrak{M}_0 \cap \mathfrak{H} = \mathfrak{B}$, and the Cartan decomposition itself has the form

$$\mathfrak{L} = \mathfrak{H} + \sum_{\alpha=1}^{p-1} \mathfrak{L}_\alpha,$$

$\mathfrak{U} = \{u\}$, $u^p = u$. Moreover, since $\mathfrak{M}_0 U = 0$, we have $\mathfrak{M}_0 = \mathfrak{B}$. Similarly, $\mathfrak{M}_1 = \mathfrak{L}_1$, $\mathfrak{M}_{-1} = \mathfrak{L}_{p-1}$, $\{a\} = \mathfrak{L}_2$, $\{b\} = \mathfrak{L}_{p-2}$, $[ab] = u$, and $\mathfrak{L} = \mathfrak{H} + \mathfrak{L}_1 + \mathfrak{L}_{p-1} + \{a\} + \{b\}$. If one takes into account the conditions $\dim \mathfrak{H} > 1$, $p > 5$, then it is not hard to see that a simple p-algebra of this type does not exist.

Lemma 2. *In any p-algebra of Lie $\mathfrak{L}$ with an abelian Cartan subalgebra $\mathfrak{H}$, $\dim \mathfrak{H} > 1$, whose semisimple component is one-dimensional, there exists an $\mathfrak{H}$-homogeneous element $a \neq 0$ such that $A^{p-1} = 0$.*

Suppose this is not so. By hypothesis,

$$\mathfrak{H} = \{u; v_1, \dots, v_m\}, \qquad u^p = u, \qquad v_i^{p^k} = 0, \qquad \mathfrak{L} = \mathfrak{H} + \sum_{\alpha=1}^{p-1} \mathfrak{L}_\alpha.$$

Since $X^{p-1} \neq 0$ for $x \in \mathfrak{L}_\alpha$, in particular $\mathfrak{L}_\alpha \neq 0$ for all α. Obviously, $e^p = h \in \mathfrak{H}$ for all $e \in \mathfrak{L}_\alpha$. Let $\beta + k\alpha = 0 \pmod p$. Then $\mathfrak{L}_\beta E^k \subset \mathfrak{H}$ and

$$\mathfrak{L}_\beta H^2 = (\mathfrak{L}_\beta E^k) H E^{p-k} = 0,$$

since $\mathfrak{H}$ is abelian. Therefore $x^p = 0$ for an arbitrary $\mathfrak{H}$-homogeneous element x. Linearization of the identity

$$(x + ty)^p = 0, \qquad x, y \in \mathfrak{L}_\alpha, \quad t \in \Phi,$$

leads to the relation $xY^{p-1} = 0$. Again using the commutativity of the algebra $\mathfrak{H}$, we choose in $\mathfrak{L}_\alpha$ a one-dimensional subspace $\{a\}$, invariant with respect to all operators from $\mathfrak{H}$. Obviously, $\mathfrak{B}A = 0$ and $\mathfrak{H}A^2 = 0$. Moreover, $\mathfrak{L}_\alpha A^{p-1} = 0$. Therefore $\mathfrak{L}A^{p-1} = \mathfrak{L}_{2\alpha} A^{p-1}$. Since $A^{p-1} \neq 0$, in the subspace $\mathfrak{L}_{2\alpha}$ one can choose a basis $b, b_1, \dots, b_r$ such that the relations

$$bA^{p-2} = -\frac{1}{\alpha} u + v, \qquad b_{iA}^{p-1} = 0, \qquad i = 1, \dots, r$$

hold. Let $f \neq 0$ be any element of the subspace $\mathfrak{L}_{i\alpha}$, $2 \le i \le p-2$. If $[fa] = 0$, then

$$if + [vf] = \left(-\frac{1}{\alpha} u + v\right) F = bA^{p-2} F = bFA^{p-2} = (bFA^{p-2-i}) A^i = hA^i = 0,$$

a contradiction, showing that $[fa] \neq 0$. Consequently,

$$\dim \mathfrak{L}_{i\alpha} A = \dim \mathfrak{L}_{i\alpha},$$

or

$$\dim \mathfrak{L}_{2\alpha} \le \dim \mathfrak{L}_{3\alpha} \le \dots \le \dim \mathfrak{L}_{(p-1)\alpha}.$$

Combining together the chains of inequalities corresponding to different roots α, we obtain the important assertion

$$\dim \mathfrak{L}_1 = \dim \mathfrak{L}_2 = \cdots = \dim \mathfrak{L}_{p-1} \geq p.$$

Without loss of generality, we may suppose that the equality $\mathfrak{B}B = 0$ holds. From what has been proved it follows that, in the whole space

$$\sum_{\alpha=1}^{p-1} \mathfrak{L}_\alpha$$

only the elements $a, b, bA, \dots, bA^{p-3}$ and their linear combinations are annihilated by the component $\mathfrak{B}$. A small additional argument shows that this is impossible.

4. In conclusion we formulate an assertion which is a consequence of Lemma 1, and also of Theorems 2–4 from the work [6].

Theorem 3. *The set of degenerate simple p-algebras of Lie of dimension $< 2p$ contains only the Witt algebra.*

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Note: Figure translations are in progress. See original paper for figures.