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Abstract

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ON THE ANALYTIC PROPERTIES OF THE SPECTRAL FUNCTION OF A SELF-ADJOINT STURM-LIOUVILLE OPERATOR

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In the present paper we investigate the analytic properties of the spectral function of the operator L, generated by the differential equation

$$-y'' + q(x)y = \lambda y \tag{1}$$

and the boundary condition

$$y(0) = 0, \tag{2}$$

where $0 \leq x < \infty$, $q(x)$ is a real locally integrable function, $\lambda = s + i\sigma$. The spectral function $\rho(s)$ of the operator L is defined only on the real axis s and, generally speaking, cannot be continued into the complex plane. R. G. showed ([1,2]) that for a finite potential $q(x)$ ($q(x) = 0$ for $x > a$) the derivative of the spectral function with respect to $\gamma = \sqrt{\lambda}$ is a meromorphic function of γ and has poles $\gamma_1, \gamma_2, ...$ ($\bar{\gamma}_1, \bar{\gamma}_2, ...$) in the lower (in the upper) half-plane, and the $\{\varphi(x, \gamma_n)\}$, where $\varphi(x, \lambda)$ is the solution of equation (1) with the initial conditions

$$\varphi(0, \lambda) = 0, \qquad \varphi'(0, \lambda) = 1, \tag{3}$$

form a complete maximal system of functions in $L_2(0, 2a)$.

In the present paper it is shown that for a potential $q(x) = -x^\alpha + p(x)$, where $p(x) = 0$ for $x > a$, the derivative $k(\lambda)$ of the spectral function of the operator L is a meromorphic function of λ, and the behavior of $k(\lambda)$ in the complex plane is studied in detail. With the aid of the results obtained, the asymptotics of the solution $\psi(x, t)$ of the Cauchy problem is found:

$$i\frac{\partial\psi(x,t)}{\partial t} = -\frac{\partial^2\psi(x,t)}{\partial x^2} - [x^\alpha - p(x)]\psi(x,t), \tag{4}$$

$$\psi(0, t) = 0, \qquad \psi(x, 0) = f(x) \quad \text{for large } t. \tag{5}$$

1. We first investigate the spectral function of the operator L_α^0, generated by the differential equation (1) with $q(x) = -x^\alpha$, where $0 < \alpha \leq 2$, i.e. by the equation

$$-y'' - x^\alpha y = \lambda y \tag{6}$$

and the boundary condition (2).

Theorem 1. 1) The derivative $k_\alpha^0(\lambda)$ of the spectral function of the operator L_α^0 is a meromorphic function of the complex variable λ; 2) the poles $\lambda_1^0, \lambda_2^0, ...$ $(\bar\lambda_1^0, \bar\lambda_2^0, ...)$ of the function $k_\alpha^0(\lambda)$ lie on the ray $|\lambda| e^{\frac{2\pi i}{\alpha+2}}$ $\left(|\lambda| e^{-\frac{2\pi i}{\alpha+2}}\right)$ and

$$|\lambda_n| \sim \left[\frac{\pi\alpha\Gamma(3/2+1/\alpha)}{\Gamma(3/2)\Gamma(1/\alpha)} n\right]^{\frac{2\alpha}{\alpha+2}} \quad \text{as } n \text{ is large;}$$

3) $k_\alpha^0(\lambda) e^{-N\sqrt{\lambda}\, i} \to 0$ on any ray $|\lambda| e^{i\theta}$, $2\pi/(2+\alpha) < \theta < 2(\alpha+1)\pi/(2+\alpha)$, as $|\lambda| \to \infty$ and for any $N > 0$ (one can prove that in the angle $2\pi/(2+\alpha) + \varepsilon \leq \theta \leq 2(\alpha+1)\pi/(2+\alpha) - \varepsilon$, where ε is any positive number smaller than $\alpha\pi/(\alpha+2)$, the function $k_\alpha^0(\lambda) e^{-iN\sqrt{\lambda}}$ tends to zero uniformly as $|\lambda| \to \infty$); 4) outside the region R, formed by the half-lines $|\lambda| e^{\pm\frac{1+\alpha}{2+\alpha}\pi i} \pm 1$, the inequality

$$|k_\alpha^0(\lambda)| < c|\sqrt{\lambda}|,$$

holds, where c is a certain constant.

Proof. It is known [3] that the derivative $k_\alpha^0(\lambda)$ of the spectral function of the operator L_λ^0 is determined as follows:

$$k_\alpha^0(s) = \lim_{\delta\to 0} \frac{1}{2i}\left[\frac{f_\alpha'(0, s+i\delta)}{f_\alpha(0, s+i\delta)} - \overline{\frac{f_\alpha'(0, s+i\delta)}{f_\alpha(0, s+i\delta)}}\right], \tag{7}$$

where $f_\alpha(x,\lambda) \in L_2(0,\infty)$ for every value of λ for which $\operatorname{Im}\lambda > 0$, and the function $f_\alpha(x,\lambda)$ is a solution of equation (6). Obviously, in order to prove assertion 1), it is enough to prove that the function $m_\alpha'(\lambda) = f_\alpha'(0,\lambda)/f_\alpha(0,\lambda)$ is a meromorphic function. To this end, consider the equation $-y'' - e^{i\varphi} x^\alpha y = \lambda y$, where $0 < \varphi \leq \pi$. It is known [4] that this equation has a solution $f_\alpha(x,\lambda,\varphi) \in L_2(0,\infty)$ for every value of φ, $0 < \varphi \leq \pi$, and for every λ, and moreover $m_\alpha^\varphi(\lambda) = f_\alpha'(0,\lambda,\varphi)/f_\alpha(0,\lambda,\varphi)$ is a meromorphic function of λ. It is not hard to see that

$$m_\alpha^\varphi(\lambda) = e^{-\frac{\pi-\varphi}{2+\alpha} i} m_\alpha^\pi\left(\lambda e^{-\frac{2(\pi-\varphi)}{2+\alpha} i}\right).$$

One can show that the function $f_\alpha(x,\lambda,\varphi)/f_\alpha(0,\lambda,\varphi)$ tends to the function $f_\alpha(x,\lambda)/f_\alpha(0,\lambda)$ uniformly in x, $0 \le x < \infty$, as $\varphi \to 0$, and for every λ in the lower half-plane. Then we obtain that the function

$$m'_\alpha(\lambda) = e^{\frac{\pi}{2+\alpha}i} m_\alpha^\pi\left(\lambda e^{-\frac{2\pi}{2+\alpha}i}\right)$$

is meromorphic, which proves assertion 1). Assertion 2) follows from the fact that the function $m_\alpha^\pi(\lambda)$ has poles $\{\mu_n\}$ only on the real axis and

$$\mu_n \sim \left[\frac{\pi\alpha\Gamma(3/2+1/\alpha)}{\Gamma(3/2)\Gamma(1/\alpha)}\, n\right]^{\frac{2\alpha}{\alpha+2}}$$

for large n. To prove assertion 3), note that the function $k_\alpha^0(\lambda)$ is an entire function of order of growth $(\alpha+2)/2\alpha$ (the proof of this fact is omitted here), and, as is known [(5)],

$$e^{-N\sqrt{|s|}}\left|k_\alpha^0(s)\right|^{-1} \to \infty$$

as $s \to -\infty$ and for every $N > 0$. Therefore, if the function $e^{+i\sqrt{\lambda}N}\,[k_\alpha^0(\lambda)]^{-1}$ were bounded on some rays $|\lambda|e^{i(\pi\pm\theta)}$, where $|\theta| < \pi\alpha/(\alpha+2)$, then by the Phragmén-Lindelöf theorem [(6)] it would follow that this function is bounded also on the negative real axis, which contradicts the condition $e^{+i\sqrt{s}N}|k_\alpha^0(s)|^{-1} \to \infty$ as $s \to \infty$. Thus assertion 3) is proved. Since [(3)] $m_\alpha^\pi(\lambda) \to i\sqrt{\lambda}$ as $|\lambda| \to \infty$ outside the domain $R_1\{|\operatorname{Im}\lambda| < 1,\ \operatorname{Re}\lambda > -1\}$, it follows that $m(\lambda) \to i\sqrt{\lambda}$ as $|\lambda| \to \infty$ and $\lambda \in R$. Hence assertion 4) follows. Thus the theorem is proved.

2. We can now pass to the study of the spectral function of the operator L_α, generated by the differential equation (1) with $q(x) = -x^\alpha + p(x)$, where $0 < \alpha \le 2$, and $p(x)$ is a finite function ($p(x) = 0$ for $x > a$, and moreover $p(x) \sim c(a-x)^l$ as $x \to a$, where $l \ge 0$ and c are fixed numbers), with the boundary condition (2).

Theorem 2. 1) The derivative $k_\alpha(\lambda)$ of the spectral function of the operator L_α admits an analytic continuation to the complex plane and is a meromorphic function of the complex variable λ; 2) the function $k_\alpha(\lambda)$ has in the upper half-plane an infinite number of poles $\lambda_1, \lambda_2, \ldots$ such that:

a) $\operatorname{Im}\lambda_n \to \infty$ as $n \to \infty$; b) the distance from the ray $|\lambda|e^{\frac{\pi}{2+\alpha}i}$ of those λ_n for which $\arg\lambda_n \ge \varepsilon > 0$ tends to zero as $n \to \infty$; c) the argument of those λ_n for which $\arg\lambda_n < \varepsilon$ tends to zero as $n \to \infty$, where ε is any fixed number smaller than $2\pi/(2+\alpha)$; 3) on every ray $|\lambda|e^{i\theta}$, where $2\pi/(2+\alpha) < \theta < 2(1+\alpha)\pi/(2+\alpha)$,

$$k_\alpha(\lambda)e^{-i\sqrt{\lambda}N} \to 0$$

as $|\lambda| \to \infty$ and for every $N > 0$, and moreover in the angle $2\pi/(2+\alpha)+\varepsilon \le \theta \le 2(1+\alpha)\pi/(2+\alpha)-\varepsilon$, where ε is any positive number smaller than $\alpha\pi/(2+\alpha)$, the function $k_\alpha(\lambda)e^{-i\sqrt{\lambda}N} \to 0$ uniformly as $|\lambda| \to \infty$.

The theorem can be proved by using the following facts.

If $F_\alpha(x,\lambda)$ is the solution of equation (1) with $q(x) = -x^\alpha + p(x)$, coinciding with $f_\alpha(x,\lambda)$ for $x > a$, then there exists a function $A(x,t)$ such that,

that is,

$$F_\alpha(x,\lambda) = f_\alpha(x,\lambda) + \int_x^{2a-x} A(x,t) f_\alpha(t,\lambda)\, dt, \tag{8}$$

$$k_\alpha(\lambda) = k_\alpha^0(\lambda) \frac{1}{\eta_1(\lambda)\eta_2(\lambda)}, \tag{9}$$

where

$$\eta_1(\lambda) = 1 + \int_0^{2a} A(0,t) \frac{f_\alpha(t,\lambda)}{f_\alpha(0,\lambda)}\, dt \tag{10}$$

and $\eta_2(s) = \overline{\eta_1(s)}$ for real s.

3. We shall call the poles $\lambda_1, \lambda_2, \ldots$ $(\bar\lambda_1, \bar\lambda_2, \ldots)$ of the function $k_\alpha(\lambda)$ the complex eigenvalues, and the corresponding functions $\varphi(x,\lambda_n)$ $(\overline{\varphi(x,\lambda_n)})$, where $\varphi(x,\lambda)$ is the solution of equation (1) for $q(x) = -x^\alpha + p(x)$ with initial conditions (3), the complex eigenfunctions of the operator L_α. For large x, the functions $\varphi(x,\lambda_n)$ $(n = 1,2,\ldots)$ grow like $x^{-\alpha/4}\exp\{|\operatorname{Im}\lambda_n| x^{1-\alpha/2}\}$ (for $\alpha = 2$, the growth of $\varphi(x,\lambda_n)$ is polynomial), and nevertheless the following theorem holds.

Theorem 3. The systems of functions $\{\varphi(x,\lambda_n)\}$ and $\{\overline{\varphi(x,\lambda_n)}\}$ form on $(0,\infty)$ a biorthogonal system of functions in the following sense:

$$\lim_{\varepsilon\to+0} \int_0^\infty e^{-\varepsilon x} \varphi(x,\lambda_n)\varphi(x,\lambda_m)\, dx = \frac{1}{2i\, \operatorname{Res} k_\alpha(\lambda_n)}\, \delta_{nm}. \tag{11}$$

Denote by K^2 the set of finite functions $f(x)$ from $L_2(0,\infty)$ such that $f(0) = 0$ and $-f'' - (x^\alpha - p(x))f \in L_2(0,\infty)$. With the aid of Theorem 2 one can prove the following theorem.

Theorem 4. If $f(x) \in K^2$, then for any $x \in (0,H)$

$$f(x) = 2i \sum_{m=1}^{n} \hat f(\lambda_m)\varphi(x,\lambda_m)\operatorname{Res} k(\lambda_m) + \frac{1}{\pi}\int_{C_n} \hat f(\lambda)\varphi(x,\lambda)k_\alpha(\lambda)\, d\lambda, \tag{12}$$

where $\hat{f}(\lambda)$ is the $\varphi(x,\lambda)$-Fourier transform of the function $f(x)$; C_n is a contour enclosing only the poles $\lambda_1, \lambda_2, \dots, \lambda_n$ of the function $k_\alpha(\lambda)$ and asymptotically parallel to the real axis, and H is any positive number.

This result may be applied to finding the asymptotics of the solution $\psi(x,t)$ of the Cauchy problem (4)–(5) for large t.

Theorem 5. If $f(x) \in K^2$, then for any $x \in (0,H)$ the solution $\psi(x,t)$ of the Cauchy problem (4)–(5) for $q(x) = -x^\alpha + p(x)$ has the form

$$\psi(x,t) = 2i \sum_{m=1}^{n} \hat{f}(\lambda_m)\varphi(x,\lambda_m)e^{i\lambda_m t} \operatorname{Res} k_\alpha(\lambda_m) + \frac{1}{\pi}\int_{C_n} \hat{f}(\lambda)\varphi(x,\lambda)e^{i\lambda t}k_\alpha(\lambda)\,d\lambda, \tag{13}$$

where $\hat{f}(\lambda)$, C_n, and H are the same as in Theorem 4, and as $t \to \infty$ it has the asymptotics

$$\psi(x,t) \sim 2i\hat{f}(\lambda_1)\varphi(x,\lambda_1) \operatorname{Res} k(\lambda_1)e^{i\lambda_1 t}, \tag{14}$$

where λ_1 is the pole of the function $k_\alpha(\lambda)$ closest to the real axis.

We now consider the case $p(x) \equiv 0$. Then it turns out that Theorem 5 can be strengthened.

Theorem 6. If $f(x) \in K^2$, then for any $x \in (0,H)$ and $t > 0$ the solution $\psi(x,t)$ of the Cauchy problem (4)–(5) for $q(x) = -x^\alpha$ has the form

$$\psi(x,t) = 2i \sum_{n=1}^{\infty} \hat{f}(\lambda_n^0)\varphi_0(x,\lambda_n^0) \operatorname{Res} k_\alpha^0(\lambda_n^0)e^{i\lambda_n^0 t}, \tag{15}$$

moreover, the series (15) converges absolutely and $\psi(x,t) \to f(x)$ as $t \to 0$, where $\varphi_0(x,\lambda)$ is the solution of equation (1) for $q(x) = -x^\alpha$ with the initial conditions (3), and $\hat{f}(\lambda)$ is the φ_0-Fourier transform of the function $f(x)$.

4. Here we continue the investigation of the analytic properties of the spectral function of the operator L generated by a more general potential $q(x)$.

Theorem 7. If L is the operator generated by the differential equation (1) with potential $q(x) = -x^\alpha + p(x)$, where $0 < \alpha < 2$ and

$$\int_0^\infty |p(x)| \exp[\varepsilon x^{1-\alpha/2}]\,dx < \infty$$

(for $\alpha = 2$ we assume that $p(x)$ satisfies the condition

$$\int_0^\infty |p(x)| x^\varepsilon \, dx < \infty)$$

for some positive ε, and with boundary condition (2), then the derivative $k_\alpha(\lambda)$ of the spectral function of the operator L admits an analytic continuation into the strip

$$|\operatorname{Im}\lambda| < 2\varepsilon(1 - \alpha/2)$$

($|\operatorname{Im}\lambda| < 2\varepsilon$ for $\alpha = 2$), and in this strip the function $k_\alpha(\lambda)$ can have only poles.

It follows from this theorem that if

$$\int_0^\infty |p(x)| \exp[\varepsilon x^{1-\alpha/2}] \, dx < \infty$$

for every $\varepsilon > 0$, then the function $k_\alpha(\lambda)$ is a meromorphic function of λ. Let us note that for $\alpha = 0$ similar results have long been known (see, for example, (7)).

Theorem 8. If L is the operator generated by the differential equation (1) with a potential $q(x)$ such that: 1) $q(x)$ decreases monotonically to $-\infty$ as $x \to \infty$; 2) $q(z)$ is a holomorphic function of $z = x+iy$ in the domain $R_3\{|z| > x_0, \arg z < \varepsilon\}$, where x_0 and ε are some fixed positive numbers; 3) $\operatorname{Im} e^{2i\varphi} q(xe^{i\varphi}) \to \infty$ as $x \to \infty$ and for every $0 < \varphi \leq \varepsilon$; 4) $|q'(z)| = O(|q(z)|^\beta)$, where $0 < \beta < 3/2$, and $|q'(z)| = O(|q(z)|')$; 5) $|q''(z)| = O(|q(z)|'')$; 6)

$$\int^\infty \frac{dx}{\sqrt{|q(x)|}} = \infty,$$

and with boundary condition (2), then the derivative $k(\lambda)$ of the spectral function of the operator L is a meromorphic function of λ.

It is clear that this theorem generalizes assertion 1) of Theorem 2 to a more general potential. The proof of this theorem is carried out analogously to the proof of assertion 1) of Theorem 1.

In conclusion we note that the results of the present work carry over to the case of the operator $L_{\alpha l}$, generated by the differential equation

$$-y'' - \left[x^\alpha - \frac{l(l+1)}{x^2} - p(x)\right] y = \lambda y,$$

where $l \geq 1/2$, and with a bounded boundary condition at zero.

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Note: Figure translations are in progress. See original paper for figures.