ON THE RANKS OF SYSTEMS OF SETS AND THE DIMENSION OF SPACES

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ON THE RANKS OF SYSTEMS OF SETS AND THE DIMENSION OF SPACES

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The present paper is devoted mainly to the study of the concept of the rank of a system of sets and of the connection of this concept with the dimension of a space. Alongside the concept of the rank of a system of sets in the sense of Nagata, it proves useful to consider ranks defined somewhat differently, in particular as was done by me earlier in ([3]).

In § 1 the most general results are given; among them the most important—the characteristic of the dimension of an arbitrary topological space—is given by Theorem 1.4.

The merits of metric spaces and bicompacts make it possible to prove stronger results for them, collected in § 2.

Finally, the case of weakly countably dimensional and countably dimensional spaces is treated separately in § 3. The theory developed there makes it possible to prove the invariance of the class of arbitrary weakly countably dimensional spaces and metric countably dimensional spaces under open, continuous, finite-to-one mappings.

I note that in this paper all coverings are assumed to be open, and the dimension of a space is everywhere the dimension defined by means of coverings. By weakly countably dimensional spaces we mean spaces representable as the sum of a countable set of their closed finite-dimensional subspaces, and by countably dimensional spaces—those representable as the sum of a countable number of their zero-dimensional subspaces. Finally, k will always denote a certain positive integer.

Let us give the basic definitions. Nagata calls two sets dependent if one of them is contained in the other. A system of sets is called dependent if it contains dependent sets; otherwise it is called independent.

Definition 0.1 (Nagata). We shall say that the rank of a system of sets γ at a point x does not exceed (the integer) k, if the system of any $k+1$ elements of the system γ containing the point x is dependent.

If the rank of the system γ at every point of the space does not exceed k, then one says that the rank of γ does not exceed k. To denote the rank of the system

γ at a point x and simply the rank of the system γ, we shall use, respectively, the notations: $r_x\gamma$, $r\gamma$.

We shall also need in what follows the notions of the small and large rank of a system (at a point and simply), which we shall denote as follows: $mr_x\gamma$, $br_x\gamma$, $mr\,\gamma$, $br\,\gamma$.

Definition 0.2. We put $mr_x\gamma \leq k$ if and only if among any $k+1$ elements of the system γ containing the point x, there is one contained in the sum of the others.*

Definition 0.3. We put $br_x\gamma \leq k$ if and only if the system decomposes into k systems of rank 1 at the point x.

Obviously, $mr_x\gamma \leq r_x\gamma \leq br_x\gamma$, if they are defined. Moreover, from $mr_x\gamma = 1$ it follows that $br_x\gamma = mr_x\gamma = r_x\gamma = 1$.

* Obviously, one could correspondingly modify the notion of a dependent system of sets.

Definition 0.4. We shall write $\operatorname{loc} r_x\gamma \leq k$ if there exists a neighborhood Ox of the point x under consideration such that the system of any $k+1$ elements of the system γ, each of which contains the point x and is itself entirely contained in Ox, is dependent.

Analogously to the definition of $r\gamma$, and in accordance with Definitions 0.2, 0.3, and 0.4, $mr\gamma$, $br\gamma$, and $\operatorname{loc} r\gamma$ are defined for an arbitrary system of sets γ.

§ 1. General results

Theorem 1.1. A T_i-space possessing a base of rank 1 is normal.

Theorem 1.2. In order that a weakly paracompact space be paracompact, it is necessary and sufficient that into every covering of this space one can inscribe a covering decomposing into a countable set of systems of rank 1.

Theorem 1.3. In order that $\dim X < k$, where X is an arbitrary bicompactum, it is necessary and sufficient that into every covering of the space X one can inscribe such a covering γ that $mr\gamma \leq k$.

In the general case the following holds:

Theorem 1.4*. In order that $\dim X < k$, where X is an arbitrary normal space, it is necessary and sufficient that into every covering of the space X one can inscribe a covering of rank $\leq k$.

Theorem 1.5. If in a space X there exists a base of rank $\leq k$, then $\dim X' < k$ and $\operatorname{ind} X' < k$ for an arbitrary subspace X' of the space X.

§ 2. The cases of metric spaces and bicompacta

Theorem 2.1. A bicompactum possessing a base of rank 1 is metrizable.

It is easy to construct an example of a normal space possessing a base of rank 1 and not metrizable. Take all transfinite numbers from the beginning up to some uncountable one inclusive. At the endpoint we define the topology in the natural way by means of the order, and at the remaining points let the space be discrete. This is the desired space. Note that if the first uncountable transfinite is chosen as the endpoint, then the space constructed will be finally compact.

However, in the general case of normal spaces the following is true:

Theorem 2.2. In order that a normal space X be homeomorphic to some metric space of dimension $< k$, it is necessary and sufficient that in X there exist a uniform base (see (1)) and a base B such that $brB \leq k$.

A close, but not identical, result to Theorem 2.2 was proved in ([3]), Theorem 1.

Theorem 2.3. In order that a compactum have dimension $< k$, it is necessary and sufficient that in it there exist a base B such that $mrB \leq k$.

I do not know whether the analogous assertion is true for arbitrary metric spaces.

Theorem 2.4.** In order that the dimension of a metric space X be less than k, it is necessary and sufficient that in X there exist a base B such that $\operatorname{loc} rB \leq k$.

As follows from Theorem 2.2, in every metric space of dimension $< k$ there exists a base B such that $rB \leq k$ and even $brB \leq k$.

* However, in the set of irrational numbers on an interval one can find a base of rank 1 from which it is impossible to choose any even point-finite covering of this set ([2]).

** Theorems 2.2 and 2.4 strengthen in both directions the following result of Nagata: a metric space has dimension $< k$ if and only if there exists in X a base of rank $\leq k$.

§ 3. Weakly Countably Dimensional and Countably Dimensional Spaces

Theorem 3.1. *In order that a metric space be countably dimensional, it is necessary and sufficient that there exist in it a base B, decomposing into a countable set of systems of rank 1 in such a way that at each point of the space some one of these systems forms a base.*

In this case it is natural to say that $\operatorname{br} B = \aleph_0$. The whole nontrivial part of Theorem 3.1 is contained in Theorem 2.2.

In connection with the classification of metric spaces according to the cardinality of the set of zero-dimensional summands into which it can be decomposed, the following is of interest:

Theorem 3.2. *Every metric space can be covered by a set of cardinality $\leq c$ of its closed zero-dimensional subspaces.*

If one assumes that $\aleph_1 = c$, then from Theorem 3.2 it follows:

Theorem 3.3. *Let X be an arbitrary metric space. There exists a sequence $\{X_\alpha\}$ of subspaces of the space X, ordered according to the first uncountable ordinal type ω_1, such that:*

1) $X_{\alpha'} \subset X_{\alpha''}$ for $\alpha' < \alpha''$;
2) $\dim X_\alpha = 0$ for $\alpha < \omega_1$;
3) $X = \bigcup\limits_{\alpha<\omega_1} X_\alpha$.

Let us return to weakly countably dimensional and countably dimensional spaces.

Theorem 3.4. *In a metric space X the following conditions are equivalent:*

1) X is weakly countably dimensional, i.e. $X = \bigcup\limits_i X_i$, where the X_i are closed in X and finite-dimensional.
2) In X there exists a refining sequence of locally finite covers $\{U_k\}$ such that, for $j > i$, the cover U_j is inscribed with its closure in the cover U_i.
3) In X there exists a base B, decomposing into a sum $B = \bigcup\limits_i B_i$ of a countable set of its subsystems of rank 1 in such a way that their bodies form a point-finite cover of the space X.
4) In X there exists a base B such that $r_x B < \infty$ for $x \in X$.
5) In X there exists a base B such that $\operatorname{loc} r_x B < \infty$ for $x \in X$.

The equivalence of conditions 1), 4), and 5) means an answer to Nagata' s question: does every metric space have a base of point-finite rank?

For compacta Theorem 3.4 admits the following strengthening:

Theorem 3.5. *In order that a compactum X be weakly countably dimensional, it is necessary and sufficient that there exist in X a base B such that $\operatorname{mr}_x B < \infty$ for $x \in X$.*

With the aid of Theorem 3.4 one proves:

Theorem 3.6. *A metric space which is an open, continuous, and finite-to-one image of a countably dimensional metric space is itself countably dimensional.*

To prove the analogous theorem for weakly countably dimensional spaces we shall need

Definition 3.1. We shall say that a numerical function $\varphi(x)$, defined on a space X, bounds the dimension of this space if into every (finite) cover of this space one can inscribe a cover whose multiplicity function on the whole space does not exceed the function $\varphi(x)$.

The following general theorem seems to me interesting:

Theorem 3.7. *A normal space is weakly countably dimensional if and only if there exists on it a function bounding its dimension.*

It follows easily from Theorem 3.7:

Theorem 3.8. *An open, continuous, and finite-to-one image of a weakly countable-dimensional space is again a weakly countable-dimensional space.*

From well-known results of Hurewicz it follows that, for closed finite-to-one mappings in the case of metric spaces, a theorem analogous to Theorem 3.6 is true, whereas an analogue of Theorem 3.8 is not true.

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Note: Figure translations are in progress. See original paper for figures.