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Abstract

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ON THE CONSTRUCTION OF DIFFERENCE EQUATIONS IN THE APPROXIMATE SOLUTION OF THE EULER–POISSON–DARBOUX EQUATION

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The method proposed in ([1]) for constructing difference equations for solving the axisymmetric Dirichlet problem for the Laplace equation by the method of nets is applied in the present paper to the construction of difference equations in solving the analogous problem for the Euler–Poisson–Darboux equation:

$$\Delta_k u = \frac{k}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1}$$

In particular, in the case of a square net with step h, 5-point difference equations are constructed for an arbitrary node of the net domain with an error of order h^3. Written for an interior node, these equations have an error of order h^4. Also constructed are 9-point difference equations for interior nodes of the net with an error of order h^8 for nodes not lying on the axis of symmetry ($k \neq -2\gamma$), and with an error of order h^6 for nodes on the axis. These 9-point equations for $k = 1$ and $k = -1$ coincide with the equations obtained by us earlier, respectively in papers ([1,2]). A 9-point difference equation with an error of order h^6, applicable to interior nodes not lying on the axis of symmetry, was obtained by another method in ([3]).

1°. Preliminary remarks. Suppose it is required to find a solution $u(r, z)$ of equation (1) in a domain G of the r, z plane, bounded by a curve Γ, with the prescribed boundary condition $u|_\Gamma = \varphi$. We shall assume that the solution of the problem posed has in the domain G continuous and bounded derivatives up to the order needed by us. Cover the domain G by an arbitrary net. Denote an arbitrary node by $\alpha_0 = \alpha_0(r_0, z_0)$.

Following ([1]), suppose that the solution $u(r, z)$ of equation (1) in a neighborhood of the point α_0 of the domain G can be represented in the form

$$u(r, z) = a_{0,0}\Phi_0^{(k)}(r, z) + \sum_{n=1}^{\infty}\left[a_{n-1,1}\Phi_{2n-1}^{(k)}(r, z) + a_{n,0}\Phi_{2n}^{(k)}(r, z)\right], \tag{2}$$

where $\Phi_0^{(k)}(r,z) \equiv 1$, $\Phi_{2n-1}^{(k)}(r,z)$, $\Phi_{2n}^{(k)}(r,z)$ are linearly independent functions satisfying equation (1) and the same conditions as in ([1,2]), while the coefficients $a_{n,0}$ $(n = 0, 1, ...)$ and $a_{n-1,1}$ $(n = 1, 2, ...)$ are determined from relations analogous to (3), (4) of ([1]).

Using formula (2), one can compose ([1]) difference equations approximately replacing equation (1), differing depending on the chosen system of functions $\Phi^{(k)}(r,z)$, the net, and the number of nodes involved.

2°. 5-point difference equations for arbitrary nodes in the case of a square net. We shall use the following systems of functions $\Phi^{(k)}(r,z)$ (we write out the first five functions):

$$\Phi_0^{(k)}(r,z) = P_0^{(k)}, \qquad \Phi_1^{(k)}(r,z) = P_1^{(k)},$$

$$\Phi_2^{(k)}(r,z) = r_0^k P_2^{(k)} + \frac{r_0}{k-1} P_0^{(k)}, \qquad \Phi_3^{(k)}(r,z) = r_0^k P_3^{(k)} + \frac{r_0}{k-1} P_1^{(k)},$$

$$\Phi_4^{(k)}(r,z) = P_4^{(k)} - \frac{r_0^{k+1}}{k+1} P_2^{(k)} - \frac{r_0^2}{2(k-1)} P_0^{(k)},$$

$$P_\sigma^{(k)} = P_\sigma^{(k)}(r, z - z_0), \qquad P_0^{(k)}(r,z) = 1, \qquad P_1^{(k)}(r,z) = z,$$

$$P_2^{(k)}(r,z) = \frac{1}{(1-k)r^{k-1}}, \qquad P_3^{(k)}(r,z) = \frac{z}{(1-k)r^{k-1}},$$

$$P_4^{(k)}(r,z) = \frac{r^2}{2(k+1)} - \frac{z^2}{2}$$

in the case $k = 0, \pm 2, \pm 3, ...$;

$$\Phi_0^{(-1)}(r,z) = P_0^{(-1)}, \qquad \Phi_1^{(-1)}(r,z) = P_1^{(-1)},$$

$$\Phi_2^{(-1)}(r,z) = \frac{1}{r_0} P_2^{(-1)} - \frac{r_0}{2} P_0^{(-1)}, \qquad \Phi_3^{(-1)}(r,z) = \frac{1}{r_0} P_3^{(-1)} - \frac{r_0}{2} P_0^{(-1)},$$

$$\Phi_4^{(-1)}(r,z) = P_4^{(-1)} - \ln r_0 \, P_2^{(-1)} + \frac{r_0^2}{4} P_0^{(-1)},$$

$$P_\sigma^{(-1)} = P_\sigma^{(-1)}(r, z - z_0), \qquad P_1^{(-1)}(r,z) = z, \qquad P_2^{(-1)}(r,z) = \frac{r^2}{2},$$

$$P_3^{(-1)}(r,z) = \frac{r^2 z}{2}, \qquad P_4^{-1}(r,z) = \frac{r^2}{2}\left(\ln r - \frac{1}{2}\right) - \frac{z^2}{2}$$

in the case $k = -1$.

In the case $k = 1$, the corresponding system of functions $\Phi^{(k)}(r,z)$ is constructed in (4).

Take the boundary node α_0 and consider the case when all four nodes nearest to it lie outside the boundary Γ (6). Let the boundary Γ intersect the straight lines joining the indicated four nodes with the node α_0 at the points $\alpha_i = \alpha_i(r_0 + k_i, z_0 + l_i)$ $(i = 1, 2, 3, 4)$, where $k_1 = t_1 h;\ k_2 = -t_2 h;\ k_3 = k_4 = 0;\ l_1 = l_2 = 0;\ l_3 = t_3 h;\ l_4 = -t_4 h;\ 0 < t_i < 1;\ h$ is the mesh step.

The difference equations that approximately replace equation (1) take, in the present case, the following form:

a) For nodes for which $r_0 \geqslant 2h$,

$$\Delta_k^h u(r_0, z_0) = \Delta_z^h u(r_0, z_0) + \Delta_{kr}^h u(r_0, z_0) = 0; \tag{3}$$

$$\Delta_z^h u(r_0, z_0) = 2\frac{t_4 u(r_0, z_0 + t_3 h) - (t_3 + t_4) u(r_0, z_0) + t_3 u(r_0, z_0 - t_4 h)}{t_3 t_4 (t_3 + t_4) h^2}; \tag{4}$$

$$\Delta_{kr}^h u(r_0, z_0) = 2E_k^{-1}\left[\alpha_k u(r_0 + t_1 h, z_0) - \beta_k u(r_0, z_0) + \gamma_k u(r_0 - t_2 h, z_0)\right]; \tag{5}$$

$$E_k = \frac{1}{k+1}\left[(r_0 + t_1 h)^2 \alpha_k - r_0^2 \beta_k + (r_0 - t_2 h)^2 \gamma_k\right], \qquad k = 0, 1, \pm 2, \pm 3, \dots;$$

$$E_{-1} = (r_0 + t_1 h)^2 \ln(r_0 + t_1 h)\, \alpha_{-1} - r_0^2 \ln r_0\, \beta_{-1} + (r_0 - t_2 h)^2 \ln(r_0 - t_2 h)\, \gamma_{-1};$$

$$\alpha_k = \frac{1}{r_0^{k-1}} - \frac{1}{(r_0 - t_2 h)^{k-1}}, \qquad \beta_k = \frac{1}{(r_0 + t_1 h)^{k-1}} - \frac{1}{(r_0 - t_2 h)^{k-1}},$$

$$\gamma_k = \frac{1}{(r_0 + t_1 h)^{k-1}} - \frac{1}{r_0^{k-1}}, \qquad k = 0, -1, \pm 2, \pm 3, \dots;$$

$$\alpha_1 = \ln\frac{r_0 - t_2 h}{r_0}, \qquad \beta_1 = \ln\frac{r_0 - t_2 h}{r_0 + t_1 h}, \qquad \gamma_1 = \ln\frac{r_0}{r_0 + t_1 h}.$$

b) For nodes for which $r_0 = h$,

$$\Delta_k^h u(h, z_0) = \Delta_z^h u(h, z_0) + \Delta_{kr}^h u(h, z_0) = 0. \tag{6}$$

Here $\Delta_z^h u(h, z_0)$ is obtained from (4) by replacing r_0 by h, while $\Delta_{kr}^h u(h, z_0)$ has the form (5) for: 1) $t_2 < 1$ and $k = 0, 1, \pm 2, \pm 3, ...$; 2) $t_2 = 1$ and $k = 0, -2, -3, ...$; 3) $t_2 \leqslant 1$ and $k = -1$. For $t_2 = 1$ and $k = 1, 2, 3, ...$, $\Delta_{kr}^h u(h, z_0)$ has the form

$$\Delta_{kr}^h u(h, z_0) = \frac{2(k+1)}{t_1(t_1+2)h^2}\,[u(h + t_1 h, z_0) - u(h, z_0)].$$

c) For nodes for which $r_0 = 0$,

$$\Delta_k^h u(0, z_0) = \Delta_z^h u(0, z_0) + \Delta_{kr}^h u(0, z_0) = 0, \tag{7}$$

where $\Delta_z^h u(0, z_0)$ is obtained from (4) by replacing r_0 by zero, and $\Delta_{kr}^h u(0, z_0)$ has the form

$$\Delta_{kr}^h u(0, z_0) = \frac{2(k+1)}{t_1^2 h^2}\,[u(t_1 h, z_0) - u(0, z_0)], \qquad k = 0, \pm 1, \pm 2, ...$$

Let us note that equation (3) for $k = 0$ coincides with the known difference equation obtained by L. V. Kantorovich in ([5]) (see formula (31), p. 226; see also the remark in ([6])), while the difference equations (3), (6), and (7) for $t_i = 1$ $(i = 1, 2, 3, 4)$ and $k = 1$ coincide with the equations obtained by us in ([4]).

3°. Nine-point difference equations for interior nodes in the case of a square grid.

Lemma. *If for an arbitrary function $S(r, z)$ the relation*

$$\Delta_k S(r, z) = \frac{1-k}{r}\frac{\partial Q(r, z)}{\partial r}, \tag{8}$$

holds, where $Q(r, z)$ is a harmonic function, $\Delta_1 Q(r, z) = 0$, then the function

$$T(r, z) = S(r, z) + Q(r, z) \tag{9}$$

satisfies the equation $\Delta_k T(r, z) = 0$.

In particular, if $Q(r, z)$ is a homogeneous polynomial of degree ν ([1]):

$$Q_\nu(r, z) = P_\nu^*(r, z) = \left(\sqrt{r^2 + z^2}\right)^\nu P_\nu\left(\frac{z}{\sqrt{r^2 + z^2}}\right),$$

then as $S(r,z)$ one may take, by virtue of (8), likewise a homogeneous polynomial $S_\nu(r,z)$ of the same degree ν. Consequently, the function (9) in the given case will have the form

$$T_\nu^{(k)}(r,s) = S_\nu(r,z) + Q_\nu(r,z).$$

The coefficients of the polynomials $S_\nu(r,z)$ are determined in the same way as in ([2]).

Now take the functions $\Phi_{2n-1}^{(k)}(r,z)$ and $\Phi_{2n}^{(k)}(r,z)$ in the form

$$\Phi_0^{(k)}(r,z) = 1,$$

$$\Phi_{2n-1}^{(k)}(r,z) = (-1)^{n-1} \sum_{\nu=1}^{n} \frac{r_0^{n-2\nu+1}\,\nu!\,2^{2\nu-n}}{(2\nu)!\,(n-\nu)!}\, T_{2\nu-1}^{(k)}(r, z-z_0),$$

$$\Phi_{2n}^{(k)}(r,z) = (-1)^{n} \sum_{\nu=0}^{n} \frac{r_0^{n-2\nu}\nu!\,2^{2\nu-n}}{(2\nu)!(n-\nu)!}\, T_{2\nu}^{(k)}(r, z-z_0) \qquad (n = 1, 2, \dots).$$

In this case the difference equations, constructed from 9 points for the interior nodes of a square grid with step h, have the form

$$\begin{aligned} u'(r_0,z_0) = {} & b_1 u(r_0+h,z_0) + b_2 u(r_0-h,z_0) + \\ & + b_3[u(r_0+h,z_0+h) + u(r_0+h,z_0-h)] + \\ & + b_4[u(r_0,z_0+h) + u(r_0,z_0-h)] + \\ & + b_5[u(r_0-h,z_0+h) + u(r_0-h,z_0-h)], \end{aligned} \tag{10}$$

$$\begin{aligned} u(0,z_0) = {} & (k+1)(5k+12)\rho_0 u(h,z_0) + \\ & + \frac{1}{2}(3-k)(4+k)\rho_0[u(0,z_0+h) + u(0,z_0-h)] + \\ & + \frac{1}{2}(k+1)(k+6)\rho_0[u(h,z_0+h) + u(h,z_0-h)], \end{aligned}$$

$$\frac{1}{\rho_0} = 5k^2 + 23k + 30, \qquad r_0 \geq 2h,$$

where for $r_0 = h$, $k \neq -1$.

The coefficients b_i $(i = 1, 2, \dots, 5)$ are determined as the solution of the system of linear equations

$$b_1 + b_2 + 2b_3 + 2b_4 + 2b_5 = 1,$$

$$\sum_{i=1}^{2} b_i \Phi_{2n}^{(k)}(r_0 + k_i, z_0 + l_i) + 2\sum_{i=3}^{5} b_i \Phi_{2n}^{(k)}(r_0 + k_i, z_0 + l_i) = 0,$$

$$k_1 = k_3 = l_3 = l_4 = l_5 = h, \qquad k_2 = k_5 = -h, \qquad l_1 = l_2 = k_4 = 0,$$

$$n = 1, 2, 3, 4.$$

The difference equation (10) for $r_0 = h$ and $k = -1$ was obtained in [(2)]. We note that equation (10) can apparently be considerably simplified, analogously to the way we did this in (7) for the Laplace equation with axial symmetry (the case $k = 1$).

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Note: Figure translations are in progress. See original paper for figures.