SYMMETRIES OF FRIEZE ORNAMENTS IN $(n+1)$-DIMENSIONAL SPACE

TIBERIU ROMAN

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Abstract

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SYMMETRIES OF FRIEZE ORNAMENTS IN $(n+1)$-DIMENSIONAL SPACE

TIBERIU ROMAN

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In the preceding paper ([1]), the symmetries of 4-dimensional frieze ornaments were defined and found. With the aid of the concept of antisymmetry, introduced by A. V. Shubnikov ([2]), frieze ornaments were represented geometrically in 3-dimensional space as two-color ornaments, and the structure of their groups of symmetries and antisymmetries was established. In the present paper the structure of the symmetry groups of $(n+1)$-dimensional $(n \geq 1)$ frieze ornaments is studied; the number of these groups is found for all orders 2^k $(k = 0, 1, 2, \dots, n, n+1)$ with the aid of the concept of α-symmetry of dimension $(n+1)$, which generalizes the concept of antisymmetry from ([2]).

Let transformations of $(n+1)$-dimensional Euclidean real space be given by the relation $y = Ax + a$, where A is a unitary matrix (i.e. $AA^+ = E$) and a is an $(n+1)$-dimensional vector. For such transformations we shall use the notation (A, a). It is known that the multiplication of two such transformations is carried out according to the rule

$$(A, a) \cdot (B, b) = (AB, Ab + a).$$

Definition 1. We shall call an **elementary set** a compact set $\mathfrak{m}$ of points of $(n+1)$-dimensional space $(n \geq 1)$, satisfying the following axioms: 1) $\mathfrak{m} \subset S$, where S is a sphere of the given radius; 2) $O \in \mathfrak{m}$, where O is the origin $(0, 0, \dots, 0)$; 3) $\mathfrak{m} \cap H_1$ admits no unitary transformations into itself except diagonal transformations (H_1 is the hyperplane $x_1 = 0$); 4) $\mathfrak{m}$ admits no transformation into itself of the form (E, a), where

$$a = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad a \neq 0.$$

Definition 2. A **frieze ornament** is a point set of $(n+1)$-dimensional space, obtained after transforming an elementary set $\mathfrak{m}$ by the group C_∞, consisting of translations (E, kt), where k is an integer,

$$t = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Definition 3. We shall call **symmetries of a frieze ornament** transformations of the form (A, a) that map the ornament onto itself. Two symmetries (A, a) and (B, b) are called **geometrically equivalent** if $A = B$; $a = b + kt$ (k an integer) or

$$(E, -{}^1/_4 t)(A, a)(E, {}^1/_4 t) = (B, b + kt).$$

Definition 4. **Rotations of a frieze ornament** are its symmetries of the form $(A, 0)$, and rotations with subsequent translations are symmetries of the form $(A, {}^1/_2 t)$.

The following theorems are easily proved:

Theorem 1. *The matrices of rotations are diagonal.*

Corollary. *There are altogether*

$$C^p_{n+1} = (n + 1)!/p!(n - p + 1)!$$

rotation matrices that contain p $(1 \leq p \leq n)$ elements equal to -1; there is only one matrix I, all nonzero elements of which are equal to -1.

By S^i_p $(i = 1, 2, \dots, C^p_{n+1})$ we shall denote one of the matrices that has p $(1 \leq p \leq n)$ entries equal to -1; in particular, we introduce the notation

$$S_1^{n+1} = \begin{pmatrix} 0 \dots 0 & 0 & & \\ 0 & 1 \dots 0 & 0 & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 \dots 1 & 0 & \\ 0 & 0 \dots 0 & -1 & \end{pmatrix} = \sigma.$$

Theorem 2. In every symmetry group of a frieze ornament the group C_∞ is a normal divisor.

Theorem 3. All geometrically inequivalent symmetries of frieze ornaments are: the identity transformation $1 = (E, 0)$; rotations $(S^i_p, 0)$, where $p = 1, 2, \dots, n$; $i = 1, 2, \dots, C^p_{n+1}$; rotations with subsequent translations $(S^j_p, {}^1/_2 t)$, where $p = 1, 2, \dots, n$; $j = 1, 2, \dots, C^p_n$ (since the element in the upper left corner cannot be equal to -1); inversion $(I, 0)$.

These definitions and theorems generalize the first definitions and theorems 1—3 from ([1]).

To determine the symmetry groups of $(n+1)$-dimensional frieze ornaments, new definitions and theorems are given that extend the last definition and theorem 5 from ([1]) and the theorem from ([5]):

Definition 5. An $(n+1)$-dimensional α-symmetry is a symmetry of an $(n+1)$-dimensional frieze ornament for which the element in the lower right corner of the matrix A is equal to -1. The other symmetries of an $(n+1)$-dimensional frieze ornament will be called β-symmetries.

Corollary 1. An $(n+1)$-dimensional α-symmetry may be regarded as a symmetry of an n-dimensional frieze ornament with a subsequent change of the signs of points.

Corollary 2. The β-symmetries of an $(n+1)$-dimensional frieze ornament may be regarded as symmetries of an n-dimensional frieze ornament.

Definition 6. The transformation $\bar{1} = (\sigma, 0)$ is called an **anti-identity transformation**; the transformation $\bar{A} = (\sigma, {}^1/_2 t)$ is called an **antitranslation by** ${}^1/_2 t$.

Definition 7. A set of β-symmetries of an $(n+1)$-dimensional frieze ornament which is a group is called a **generating symmetry group of n-dimensional two-color frieze ornaments** (briefly: a generating group n_2).

The direct product of a generating group n_2 with the group $\{1, \bar{1}\}$ is called a **neutral symmetry group of n-dimensional two-color frieze ornaments** (briefly: a neutral group n).

The direct product of a generating group n_2 with the group $\{1, \bar{A}\}$ is called a **pseudoneutral symmetry group of n-dimensional two-color frieze ornaments** (briefly: a pseudoneutral group n_2).

The other symmetry groups of $(n+1)$-dimensional frieze ornaments are called **mixed symmetry groups of n-dimensional two-color frieze ornaments** (briefly: mixed groups n_2).

Theorem 4. The symmetry groups of $(n+1)$-dimensional frieze ornaments are: generating groups n_2; neutral groups n; pseudoneutral groups n_2; mixed groups n_2.

Mixed groups n_2 are obtained as follows: the generating groups n_2 are decomposed into right (or left) cosets with respect to all their subgroups of index 2, and in all classes that do not contain the identity of the group, β-symmetries are replaced by the corresponding α-symmetries.

Now one can establish the main theorem of the present paper, which gives the structure of the symmetry groups of $(n+1)$-dimensional frieze ornaments and the number $(2^k)_{n+1}$ of all groups of order 2^k of geometrically inequivalent symmetries.

Theorem 5. $(n+1)$-dimensional frieze ornaments admit groups of geometrically inequivalent symmetries of order 2^k $(k = 0, 1, 2, \dots, \dots, n, n+1)$; their number is:

$$(2^0)_{n+1} = 1; \quad (2^1)_{n+1} = 3 \cdot 2^n - 2;$$

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$$(2^k)_{n+1} = \frac{(2^n-1)(2^{n-1}-1)\dots(2^{\,n-(k-2)}-1)}{(2^2-1)(2^3-1)\dots(2^k-1)}\left[(2^k+1)2^n-2^k\right], \qquad (1)$$

where $k = 2, 3, \dots, n, n+1$.

Indeed, rotations, rotations followed by translations, and inversions (Theorem 3) are elements of order 2, since $(S_p^i, 0)\cdot(S_p^i, 0) = (E, 0)$; $(I, 0)(I, 0) = (E, 0)$, while the symmetry $(S_p^i, \frac{1}{2}t)\,(S_p^i, \frac{1}{2}t) = (E, t)$ is geometrically equivalent to $(E, 0)$.

From elements of the second order one can form only groups of order 2^k of type $(2, 2, \dots, 2)$.

$$\underbrace{\qquad\qquad\qquad}_{k}$$

It is obvious that there is only one group of order $1 = 2^0$: $(E, 0)$. To determine the number of groups of geometrically inequivalent symmetries of the second order, note that they have the form $\{1, U\}$, where U is an inversion, a rotation, or a rotation followed by a translation. The number of rotations is $C_{n+1}^1 + C_{n+1}^2 + \dots + C_{n+1}^n = 2^{n+1} - 2$ (by the corollary to Theorem 1); the number of rotations followed by a translation is $C_n^1 + C_n^2 + \dots + C_n^n = 2^n - 1$. Hence it follows that the number of groups of the second order is given by the formula

$$(2^{n+1} - 2) + (2^n - 1) + 1 = 3\cdot 2^n - 2.$$

To determine the number $(2^k)_{n+1}$ of groups of geometrically inequivalent symmetries of order 2^k $(k = 2, 3, \dots, n+1)$ of $(n+1)$-dimensional frieze ornaments, Theorem 4 is applied: the number of generating groups n_2, geometrically inequivalent, of order 2^k is equal to $(2^k)_n$; the number of neutral groups n, geometrically inequivalent, of order 2^k is equal to $(2^{k-1})_n$, since these groups are direct products of generating groups n_2 (of order 2^{k-1}) with the group $\{1, I\}$; the number of pseudoneutral groups n_2, geometrically inequivalent, of order 2^k is equal to $(2^{k-1})_n$, since they are direct products of generating groups n_2 (of order 2^{k-1}) with the group $\{1, \bar{A}\}$; the number of mixed groups n_2, geometrically inequivalent, of order 2^k is equal to $(2^k - 1)(2^k)_n$, since each of the $(2^k)_n$ generating groups n_2 is decomposed into 2 right (or left) cosets with respect to $2^k - 1$ subgroups of index 2 (one subgroup has first order), one of these two classes does not contain the identity of the group. Consequently, the recurrence relation is established

$$(2^k)_{n+1} = 2^k(2^k)_n + 2(2^{k-1})_n. \qquad (2)$$

For $k=2$ this relation makes it possible to obtain the particular case of (1):

$$(2^2)_{n+1} = \frac{2^n-1}{2^2-1}\left[(2^2+1)2^n - 2^2\right].$$

Formula (1) is proved by a twofold application of the method of complete induction. It is valid for $k=2$; suppose that it is also satisfied for $k=p$; this makes it possible to compute $(2^p)_{n+1}$ and $(2^p)_n$. The rela-

relation (2) for $k=p+1$ can be written in the form

$$(2^{p+1})_{n+1} = 2^{p+1}(2^{p+1})_n + 2\frac{(2^{n-1}-1)\cdots(2^{n-p+1}-1)}{(2^2-1)\cdots(2^p-1)}\left[(2^p+1)2^{n-1} - 2^p\right]. \quad (3)$$

Since $(2^q)_2 = 0, \quad (2^q)_3 = 0$ $(q=4,5,...)$ (see (3) or (4)), it follows from (2) that

$$(2^{n+h})_n = 0. \qquad (h=1,2,...). \quad (4)$$

Replacing k by $k+1$ in (2), we obtain

$$(2^{k+1})_{n+1} = 2^{k+1}(2^{k+1})_n + 2(2^k)_n. \quad (2')$$

Formula (1), which was assumed to be valid for $k=p$, gives $(2^p)_p = 2^{p-1}$, whence, for $k=p$ and $n=p$, $(2')$ leads to $(2^{p+1})_{p+1} = 2^{p+1}(2^{p+1})_p + 2(2^p)_p$, and, taking (4) into account, $(2^{p+1})_{p+1} = 2^p$.

Consequently, relation (1) is verified for $k=p+1$ and $n=p$ (for $n<p$ it is also valid, since, by (4), $(2^{p+1})_{n+1} = 0$). Relation (3) makes it possible to compute $(2^{p+1})_{p+2}$:

$$(2^{p+1})_{p+2} = 2^{p+1}(2^{p+1})_{p+1} + 2\frac{(2^p-1)\cdots(2^2-1)}{(2^2-1)\cdots(2^p-1)}\left[(2^p+1)2^p - 2^p\right] =$$

$$= 2^{p+1}\cdot 2^p + 2\cdot 2^{2p} = 2^{2(p+1)},$$

whence it is clear that (1) is verified for $k=p+1$ and $n=p+1$.

Assume that (1) is valid for $k=p+1$ and $n=m>p$. Relation (2) gives

$$(2^{p+1})_{m+1} = 2^{p+1}(2^{p+1})_m + 2(2^p)_m. \quad (5)$$

But $(2^{p+1})_m$ and $(2^p)_m$ can be computed by means of (1) on the basis of the assumptions made above. Relation (5) makes it possible to compute $(2^{p+1})_{m+1}$, which agrees with what (1) gives. Consequently, (1) is valid for $k = p + 1$ and all n. Since (1) is valid for $k = 2$ and arbitrary n, and it has been proved that it is valid for $k = p + 1$ and all n, Theorem 5 is completely proved.

Remark. The number of groups of geometrically inequivalent symmetries, of order 2^k, of $(n+1)$-dimensional frieze ornaments is given by the following table, from which one can also compute the previously known values (see (3) or (4) and (1)) for 2-, 3-, and 4-dimensional spaces:

k/n	2	3	4
0	1	1	1
1	4	10	22
2	2	16	84
3		4	64
4			8

$$(2^0)_{n+1} = 1; \qquad (2^1)_{n+1} = 3 \cdot 2^n - 2; \qquad (2^2)_{n+1} = \frac{2^n - 1}{2^2 - 1}(5 \cdot 2^n - 4);$$

$$(2^3)_{n+1} = \frac{(2^n - 1)(2^{n-1} - 1)}{(2^2 - 1)(2^3 - 1)}(9 \cdot 2^n - 8);$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(2^k)_{n+1} = \frac{(2^n - 1)\cdots(2^{n-k+2} - 1)}{(2^2 - 1)\cdots(2^k - 1)}\left[(2^k + 1)2^n - 2^k\right];$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(2^{n-1})_{n+1} = \frac{2^{2n} - 1}{2^2 - 1}\, 2^{n-1}; \qquad (2^n)_{n+1} = 2^{2n}; \qquad (2^{n+1})_{n+1} = 2^n.$$

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Note: Figure translations are in progress. See original paper for figures.