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Abstract

Full Text

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THE SPACE OF CONVERGENT SEQUENCES

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It is known that the topological product of groups coincides with the closure of the subspace of their discrete direct product. However, the discrete direct product of groups, considered as a subspace of some topological group, cannot always be embedded homeomorphically in the corresponding topological product. Therefore the question of describing the closure of a discrete direct product of groups is a topical one. In the present note a partial solution of this question is derived from results proved for uniform spaces.

Uniform spaces will be called **similar** if the partially ordered sets of their entourages are isomorphic (([2]), p. 154). Let R_i $(i \in I)$ be similar uniform spaces, in each R_i let an element O_i be distinguished, and let Ω be the partially ordered set of their entourages. If a and b are close of order α $(\alpha \in \Omega)$, then we shall write $a \equiv b$ (α). Further, let R be the direct product of the sets R_i. Denote by π_i the projection of R onto R_i. An element $(a_i) \in R$ will be called **convergent** if, whatever $\alpha \in \Omega$ may be, $a_i \equiv O_i$ (α) for almost all i. Consider the set of convergent elements of R, and put $(a_i) \equiv (b_i)$ (α) $(\alpha \in \Omega)$ if $a_i \equiv b_i$ (α) for all i. The resulting uniform space will be called the **space of convergent sequences** of the space R and will be denoted by $\Pi^c R_i$. An element $(a_i) \in R$ will be called **finite** if $a_i = O_i$ for almost all i.

Putting $S = \Pi^c R_i$, it is easy to verify the following facts:

Proposition 1. The natural embedding of R_i in S is a uniform homeomorphism*; moreover, if all R_i are separated, then R_i is closed in S.

Proposition 2. The set of finite elements is everywhere dense in S.

Proposition 3. If all R_i are separated, then S is also separated.

Theorem 1. The space S is bicompact if and only if: 1) all R_i are bicompact; 2) whatever $\alpha \in \Omega$ may be, almost all R_i are small of order α.

Proof. If S is bicompact, then the validity of 1) follows from Proposition 1. If 2) does not hold, then there exists an infinite sequence $x^1, x^2, ...$ of elements from $R_1, R_2, ...$, such that $x^k \not\equiv O_k$ (α). Since $R_k \subset S$, $x^k \to x$. Suppose that $\pi_i(x) \neq O_i$. Then some β-neighborhood of the element $\pi_i(x)$ $(\beta \in \Omega)$ does not contain O_i. But then the β-neighborhood of the element $\pi_i(x)$ contains x^i. In view of the arbitrariness of the smallness of β, we have $\pi_i(x) = x^i$. Since $x \in S$, this is possible only for a finite set of values of i. Consequently, $\pi_i(x) = O_i$ for almost all i. But then the infinite set of elements x^k does not belong to the α-neighborhood of the element x—a contradiction. If, conversely, conditions 1) and 2) are fulfilled, then $S = R$, and the topology on S coincides with the Tychonoff topology. It remains to take into account Tychonoff' s theorem (([1]), p. 394; ([2]), p. 114).

* In ([2]) (p. 160) the term "isomorphism" is used.

Theorem 2. The space S is bicompact at the point (O_i) (i.e., has a neighborhood of the point (O_i) with bicompact closure) if and only if there exists a neighborhood $\alpha \in \Omega$ such that: 1) the closure of the α-neighborhood of O_i in each R_i is bicompact; 2) for any $\beta \in \Omega$, for almost all i the closure of the α-neighborhood of O_i in R_i is of order less than β.

We shall precede the proof by two lemmas.

Lemma 1. If U is an α-neighborhood of the point (O_i) in S, then $\overline{U} = \prod^c \pi_i(\overline{U})^*$.

Indeed, the inclusion $U \subset A$, where $A = \prod^c \pi_i(\overline{U})$, is obvious. If $a \in A$ and $\beta \in \Omega$, then there is a finite set of indices K such that $\pi_i(a) \equiv O_i(\beta)$ for $i \notin K$. Since $\pi_i(a) \in \pi_i(\overline{U})$, there exist $u^k \in U$ $(k \in K)$ such that $\pi_k(u^k) \equiv \pi_k(a)\,(\beta)$. Let $u \in R$ be such that $\pi_i(u) = \pi_i(u^i)$ for $i \in K$ and $\pi_i(u) = O_i$ for $i \notin K$. Then $u \in U$ and $a \equiv u\,(\beta)$. Consequently, $a \in \overline{U}$, and hence $A \subset \overline{U}$.

Lemma 2. If U is an α-neighborhood of the point (O_i) in S, and V is the α-neighborhood of the point O_i in R_i, then $\pi_i(\overline{U}) = \overline{V}$.

Indeed, if $x \in \pi_i(\overline{U})$, then $x = \pi_i(y)$, where $y \in \overline{U}$. If $\beta \in \Omega$, then $y \equiv u(\beta)$, where $u \in U$. Then $x \equiv \pi_i(u)(\beta)$. But $\pi_i(u) \equiv O_i(\alpha)$, i.e. $\pi_i(u) \in V$. Consequently, $x \in \overline{V}$, i.e. $\pi_i(\overline{U}) \subset \overline{V}$. Suppose now that $x \in \overline{V}$ and $\beta \in \Omega$. Then $x \equiv y(\beta)$, where $y \equiv O_i(\alpha)$. In view of Proposition 1, $x, y \in S$. But then $y \in U$. Consequently, $x \in \overline{U}$, and hence $x \in \pi_i(\overline{U})$. Thus, $\overline{V} \subset \pi_i(\overline{U})$.

Now let S be bicompact at the point (O_i). Choose an α-neighborhood U of the point (O_i) with bicompact closure. In view of Lemmas 1 and 2, $\overline{U} = \prod^c \overline{V}_i$, where V_i is the α-neighborhood of O_i in R_i, so that the validity of 1) and 2) follows from Theorem 1. The sufficiency of conditions 1) and 2) also follows from Theorem 1 with the help of Lemmas 1 and 2.

Theorem 3. If all R_i are separated and complete, then S is the completion of the set of finite elements.

First we prove two lemmas.

Lemma 3. Let $\mathfrak{F}$ be a base of a Cauchy filter ([2]), p. 168, in the set of finite elements, $\alpha \in \Omega$, and $\beta \overset{2}{<} \alpha$. Then there exists a finite set of indices K such that $\pi_i(A) \equiv O_i(\alpha)$ for all $i \notin K$, if $A \in \mathfrak{F}$ and is of order less than β.

Indeed, suppose that there is an infinite set of indices s such that $A^s \in \mathfrak{F}$, A^s is of order less than β, $a^s \in A^s$, $\pi_s(a^s) \not\equiv O_s(\alpha)$. Renumbering, if necessary, we shall have $\pi_2(a^1) = O_2$. Consequently, $a^1 \not\equiv a^2(\alpha)$. Since A^1 and A^2 are of order less than β and the intersection $A^1 \cap A^2$ is nonempty, this is impossible.

Lemma 4. If R_i are complete, then S is complete.

Indeed, let $\mathfrak{F}$ be a base of a Cauchy filter in the set of finite elements. Put $\mathfrak{F}_i = \pi_i(\mathfrak{F})$. It is clear that $\mathfrak{F}_i$ is a base of a Cauchy filter in R_i. Let x_i be the limit of this base ([2]), p. 62. Denote by x the element of R such that $\pi_i(x) = x_i$ for all i. If $x \notin S$, then there exist $\alpha \in \Omega$ and an infinite sequence of indices Λ such that $x_i \not\equiv O_i(\alpha)$ for all $i \in \Lambda$. Let $\beta \overset{3}{<} \alpha$, $A \in \mathfrak{F}$, and let A be of order less than β. In view of Lemma 3, for some $k \in \Lambda$ we shall have $\pi_k(A) \equiv O_k(\beta)$. But then $\pi_k(A) \not\equiv x_k(\beta)$, which is incompatible with $\pi_k(A) \in \mathfrak{F}_k$ (see ([2]), p. 62, Proposition 1). Thus, $x \in S$. Let again $\alpha \in \Omega$ and $\beta \overset{6}{<} \alpha$. Taking into account Lemma 3 and the relation $x \in S$, we find a finite set of indices K such that, whatever the set A from $\mathfrak{F}$ of order less than β may be, for all $i \notin K$ one has

* By $\overline{M}$ is denoted the closure of the set M.

$x_i = O_i(\beta)$ and $\pi_i(A) \equiv O_i(\beta)$. Since $\mathfrak{F}_i \to x_i$, there is a $B \in \mathfrak{F}$ such that $\pi_i(B) \equiv x_i(\beta)$ for all $i \in K$. Since B is of order β, $B \equiv x(\alpha)$. Hence $\mathfrak{F} \to x$. It remains to take into account Proposition 2 and ([2]), p. 170, Proposition 7.

In view of Propositions 2 and 3, the validity of Theorem 3 follows from Lemma 2 and ([2]), p. 175, Proposition 9.

Let G be a topological group. In a natural way it may be regarded as a uniform space (([2]), p. 224). A system $\mathfrak{H} = \{H_i, i \in I\}$ of closed subgroups of the group G will be called **independent** if each finite subsystem of it generates in G a subgroup that is the topological direct product of the corresponding subgroups from $\mathfrak{H}$. Turn H_i into the uniform space $\hat{H}_i$, taking as a system of neighborhoods of the identity the system of sets of the form $p_i(U)$, where p_i is the projection $H = \sum H_i{}^*$ onto H_i, and U is a neighborhood of the identity in H. The uniform space $S = \prod^c \hat{H}_i$ will be called the **accompanying space** of the system $\mathfrak{H}$. The mapping $h \to (p_i(h))$ of the space $\bar{H}$ onto the set of finite elements of S will be called **natural**. We shall say that the projections p_i are **uniformly equicontinuous** if for every neighborhood U of the identity in $\bar{H}$ there is a neighborhood V of the identity in H such that $p_i(V) \subset H_i \cap U$ for all $i \in I$. It is easy to prove:

Proposition 4. *The natural mapping is one-to-one and uniformly continuous; if the group H admits a complete system of neighborhoods of the identity consisting of subgroups, and the projections are uniformly equicontinuous, then the natural mapping is a uniform homeomorphism.*

Theorem 4. *Let G be a complete separated topological group, $\mathfrak{H} = \{H_i\}$ an independent system of subgroups of G, and $H = \sum H_i$. Then the natural mapping of the group H into the accompanying space S of the system $\mathfrak{H}$ can be extended to a uniformly continuous mapping of the group $\bar{H}$ onto S; if the group H admits a system of neighborhoods of the identity consisting of subgroups,* *and the projections H onto H_i are uniformly equicontinuous, then the above-mentioned mapping of $\bar{H}$ onto S is a uniform homeomorphism.*

Proof. The space $\bar{H}$ is complete (([2]), p. 170, Proposition 6). Therefore the validity of the theorem follows from Proposition 4, Theorem 3, ([2]), p. 171, Theorem 1, and ([2]), p. 175, Proposition 9.

Theorem 5. *If the topological groups H_i are complete uniform spaces and $H = \prod^c H_i$ has a group neighborhood of zero U with bicompact closure, then H is a local direct product of the groups H_i with respect to the subgroups $\pi_i(U)$.*

For the proof it suffices to take into account Theorem 2 and the definition of a local direct product of groups (([3]), p. 9).

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* What is meant is the subgroup generated in G by the union $\bigcup H_i$; obviously, it coincides with the discrete direct product of the groups H_i.

Note: Figure translations are in progress. See original paper for figures.