ON THE INTERPRETATION OF NONSTATIONARY RELATIVISTIC HYDRODYNAMICS IN MINKOWSKI SPACE

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Abstract

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HYDROMECHANICS

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ON THE INTERPRETATION OF NONSTATIONARY RELATIVISTIC HYDRODYNAMICS IN MINKOWSKI SPACE

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We shall consider adiabatic motions of an ideal (nonviscous) gas in relativistic hydrodynamics (without taking gravitational fields into account). As was shown in ([5]), every stationary motion in relativistic hydrodynamics, in the corresponding quantities, can be interpreted as a nonrelativistic stationary motion of the corresponding gas. It is shown below that nonstationary motions in relativistic hydrodynamics can be interpreted, in the same quantities in four-dimensional space-time, as nonrelativistic stationary motions, taking the constant in Bernoulli' s equation to be equal to 0.

1. Let us consider four-dimensional space-time: x_1, x_2, x_3 are spatial rectangular Cartesian coordinates, $x_4 = ict$ is the imaginary time coordinate.

Components of 4-vectors are denoted by the index k, which takes the values $1, 2, 3, 4$. The index α takes the values $1, 2$, and 3. The interval $ds^2 = -dx_k^2$. The 4-velocity $u_k = dx_k/ds$, so that $u_\alpha = v_\alpha/c\sqrt{1 - v_\alpha^2/c^2}$, $u_4 = i/\sqrt{1 - v_\alpha^2/c^2}$; v_α is the three-dimensional velocity ([1]).

The fundamental equations—the laws of conservation of momentum and energy and the law of conservation of particles—we shall take in the form ([2],[4],[5]):

$$n\frac{d}{ds}\left(\frac{w}{n}u_k\right) = -\frac{\partial p}{\partial x_k}, \tag{1}$$

$$\frac{d}{ds}\left(\frac{w}{n}\right) - \frac{1}{n}\frac{dp}{ds} = 0, \tag{2}$$

$$\frac{\partial (n u_k)}{\partial x_k} = 0. \tag{3}$$

Here d/ds denotes the total derivative along the world line of a particle: $d/ds = u_k \partial/\partial x_k$; the thermal function w and the number of particles n are referred to unit proper volume; p is the pressure. Formula (1) contains only 3 independent equations.

Analogously to ([5]), introduce new unknown functions:

$$\frac{w}{mnc^2} cu_k = \nu_k, \qquad mn\left(\frac{mnc^2}{w}\right) = \tilde{\rho}, \qquad \frac{c^2}{2}\left(\frac{w}{mnc^2}\right)^2 = I; \tag{4}$$

here m is the rest mass of a particle, so that mn denotes the rest density, while the dimensionless quantity w/mnc^2 denotes the thermal function referred to unit rest energy. In contrast to ([5]), in the present case ν_k denotes 4-components.

In these quantities the system (1)–(3) takes the form

$$\tilde{\rho}\nu_k \frac{\partial \nu_\alpha}{\partial x_k} = \tilde{\rho}\frac{d}{ds}\nu_\alpha = -\frac{\partial p}{\partial x_\alpha}; \tag{5}$$

$$\tilde{\rho}\frac{dI}{ds} - \frac{dp}{ds} = 0; \tag{6}$$

$$\frac{\partial}{\partial x_k}(\tilde{\rho}\nu_k) = 0. \tag{7}$$

To these equations it is necessary to add the relation $u_k^2 = -1$, which in the new variables is written in the form:

$$\mathfrak{v}_k^2/2 + I = 0. \tag{8}$$

As shown in ([5]), one can introduce an auxiliary gas with pressure p (equal to the original pressure of the relativistic gas), density $\tilde{\rho}$, and thermal function of unit mass I. For this gas the entropy of unit mass $\tilde{S}$ coincides with the entropy σ/mn of unit rest mass of the original relativistic gas, so that the adiabaticity conditions for these two gases are fulfilled simultaneously.

Disregarding the meaning of x_4 and considering it as a fourth spatial coordinate, let us consider (unlike in ([5])) a fictitious four-dimensional Euclidean space x_1, x_2, x_3, x_4. In this space the system (5)–(8) describes nonrelativistic stationary adiabatic motions of an auxiliary gas with four-dimensional velocity $\mathfrak{v}_k$. Namely, (6) expresses adiabaticity, (7) is the continuity equation for stationary motion, and the three equations (5) are the projections onto the axes x_1, x_2, x_3 of Newton' s momentum equation in four-dimensional space. Instead of the fourth projection of the momentum equation, the Bernoulli equation (8) is written. In this case the constant in the Bernoulli equation is equal to 0 throughout the entire region of motion.

In the nonrelativistic limit, equation (8) turns into a trivial identity, and the preceding arguments lose their meaning.

In the indicated manner one can consider any motion in relativistic hydrodynamics, but this turns out to be essential precisely for nonstationary relativistic motions. For stationary relativistic motions, however, the need to resort to a four-dimensional space in which the Bernoulli constant is equal to 0 disappears altogether; it is sufficient to consider an auxiliary gas with a three-dimensional velocity in ordinary three-dimensional space, in which it moves in a physically natural way, as was shown in ([5]).

Let us note that analogous considerations can be applied to the relativistic motion of a point. Disregarding the meaning of x_4, in a fictitious four-dimensional space relativistic motion with 4-velocity u_k can be formally considered as a four-dimensional Newtonian motion with four-dimensional velocity cu_k, whose absolute magnitude does not change ($c^2 u_k^2 = -c^2$), since in this space all forces (equal to the Minkowski 4-forces) are perpendicular to the velocity. In particular, motion along a circle under the action of a constant "centripetal force" corresponds to relativistic hyperbolic ("uniformly accelerated") motion ([1]).

2. Let us consider one-dimensional nonstationary relativistic motions. Plane one-dimensional relativistic motions correspond to nonrelativistic plane-parallel stationary motions, and cylindrical ones to stationary axisymmetric motions. Spherical relativistic motions have no ordinary nonrelativistic analogue.

Let us examine in more detail the case of plane one-dimensional nonstationary relativistic motion, to which there corresponds a plane-parallel stationary motion of the auxiliary gas in the plane x_1, x_4. Let $\mathfrak{v}$ be the modulus of its velocity, and θ its angle with the axis x_4, so that $\mathfrak{v}_1 = \mathfrak{v} \sin\theta$, $\mathfrak{v}_4 = \mathfrak{v} \cos\theta$. From (8) and (4) it follows that $\mathfrak{v} = i\sqrt{2I} = iw/mnc$. Everywhere below i denotes the imaginary unit. Introducing the variable $\eta = i\theta$, we obtain $\mathfrak{v}_1 = \sqrt{2I}\,\mathrm{sh}\,\eta$, $\mathfrak{v}_4 = i\sqrt{2I}\,\mathrm{ch}\,\eta$, or, in the original quantities, $u_1 = \mathrm{sh}\,\eta$, $u_4 = i\,\mathrm{ch}\,\eta$, $v/c = \mathrm{th}\,\eta$.

The linear equation obtained by I. M. Khalatnikov ([4]) for isentropic one-dimensional relativistic motions is, in essence, Chaplygin' s equation (([2]), formula (108.8)) for potential plane-parallel motions of an auxiliary gas, in which one must express

express θ through η and ϑ through $\mathcal{W}/n$ in the manner indicated above and set $1 - \vartheta^2/a^2 = c^2/\omega^2$, where a and ω are the speeds of sound in the auxiliary and relativistic gases, respectively ([5]), and c is the speed of light.

Since in ([4]) a misprint crept into formula (4.13) expressing this equation, we give it below in the notation of the present paper:

$$\frac{\omega^2}{c^2}\frac{\partial^2\Phi}{\partial y^2} + \left(1 - \frac{\omega^2}{c^2}\right)\frac{\partial\Phi}{\partial y} - \frac{\partial^2\Phi}{\partial\eta^2} = 0, \tag{9}$$

where $y = \ln \mathcal{W}/n$, $\Phi = -\varphi + x_1\vartheta_1 + x_4\vartheta_4 = -\varphi + (x_1u_1 + x_4u_4)\mathcal{W}/mnc$, $\vartheta_k = \partial\varphi/\partial x_k$.

Let us apply the analogy considered above to clarify the question of in what cases, for non-isentropic adiabatic motions, one can obtain a linear equation similar to (9). This reduces to the possibility of obtaining a linear equation for the vortex plane-parallel motion of the auxiliary gas. We shall consider only the case when this is possible for an arbitrary distribution of entropy along the streamlines (i.e., over the particles in the original gas). The desired condition is connected with the requirement that the differential relations along the characteristics $d\theta \pm \sqrt{\vartheta_{pp}/\vartheta}\, dp = 0$ ([3], p. 313; it is understood that ϑ is expressed from Bernoulli' s equation through p and $\tilde{S}$) be representable by finite relations. This is possible only when ϑ_{pp}/ϑ does not depend on $\tilde{S}$ (as in potential motion). Since $\vartheta = i\mathcal{W}/mnc$, this requirement is satisfied by all media for which $(\mathcal{W}/n)_{pp}/(\mathcal{W}/n)$ does not depend on σ/n. Denote

$$(\mathcal{W}/n)_{pp} = -F^2(p)(\mathcal{W}/n). \tag{10}$$

Taking into account, from the thermodynamic identity $d(\mathcal{W}/n) = T\, d(\sigma/n) + dp/n$, that $1/n = (\mathcal{W}/n)_p$, we write (10) in the form $\mathcal{W}_p = 1 + F^2(p)\mathcal{W}^2$. This condition is satisfied, in particular, by any substance in the ultrarelativistic case, when $\mathcal{W} = 4p$. From the definition of the speed of sound ω ([2,5]) it follows that $F(p) = c/\omega\mathcal{W}$.

To obtain the desired linear equation, consider the original relativistic equations (1)–(3) in Lagrangian coordinates; as these we choose $\tau = ct = x_4/i$ and the stream function ψ of the auxiliary plane-parallel motion, since the streamlines of the auxiliary gas are the trajectories of the particles of the original gas. Define ψ by the formulas following from (3): $nu_1 = -\partial\psi/\partial\tau$, $nu_4 = i\,\partial\psi/\partial x_1$, so that $iu_1/u_4 = v/c = \operatorname{th}\eta = (\partial x_1/\partial\tau)_\psi$. The condition of adiabaticity means that σ/n depends only on ψ. As the unknown functions we choose η and p.

We take the continuity equation (3) in the form $\partial x_1/\partial\psi = i/nu_4 = 1/n \operatorname{ch}\eta$, or $\partial(\operatorname{th}\eta)/\partial\psi = \partial(1/n \operatorname{ch}\eta)/\partial\tau$. Noting that

$$\frac{1}{n} = \frac{\partial}{\partial p}\left\{\frac{\mathcal{W}}{n}\left[p, \frac{\sigma}{n}(\psi)\right]\right\},$$

we finally write it in the form

$$\frac{\partial\eta}{\partial\psi} + \frac{\operatorname{sh}\eta}{n}\frac{\partial\eta}{\partial\tau} + \frac{\mathcal{W}\operatorname{ch}\eta}{n}F^2(p)\frac{\partial p}{\partial\tau} = 0. \tag{11}$$

The momentum equation has the form $\partial(\mathcal{W}\operatorname{sh}\eta/n)/\partial\tau = -\partial p/\partial\psi$, or

$$\frac{\partial p}{\partial\psi} + \frac{\operatorname{sh}\eta}{n}\frac{\partial p}{\partial\tau} + \frac{\mathcal{W}\operatorname{ch}\eta}{n}\frac{\partial\eta}{\partial\tau} = 0. \tag{12}$$

Denoting $F(p) = A'(p)$, we obtain the equations of the characteristics of the system (11)–(12) and the conditions along them in the form

$$[\operatorname{sh}\eta \pm \mathcal{W}A'(p)\operatorname{ch}\eta]\,d\psi = n\,d\tau, \qquad \eta \pm A(p) = \text{const}. \tag{13}$$

In what follows we shall consider the case when w does not depend on σ/n, i.e. $w = w(p)$. Then from $(w/n)_p = 1/n$ it follows that $n = f(\sigma/n)\exp\{\int[w'(p) - 1]w^{-1}\,dp\}$. Let us denote $f[\frac{\sigma}{n}(\psi)] = [z'(\psi)]^{-1}$ and $1/n = z'(\psi)\xi(p)$.

In a simple wave the corresponding family of characteristics is a pencil of straight lines in the plane (z, τ) (and also in the plane x, t).

Interchanging in the system (11)–(12) the roles of the dependent variables η and p and the independent variables z and τ, we obtain the linear system

$$\frac{\partial \tau}{\partial p} = \xi(p)\operatorname{sh}\eta\,\frac{\partial z}{\partial p} - w(p)\xi(p)F^2(p)\operatorname{ch}\eta\,\frac{\partial z}{\partial \eta},$$

$$\frac{\partial \tau}{\partial \eta} = \xi\,\operatorname{sh}\eta\,\frac{\partial z}{\partial \eta} - w\xi\,\operatorname{ch}\eta\,\frac{\partial z}{\partial p}. \tag{14}$$

Noting that $F^2 = -\xi'/w\xi = c^2/(\omega w)^2$ and introducing the variable $y = \ln[w(p)\xi(p)]$, from this system we obtain for $z(\eta, y)$ the linear equation, analogous to (9):

$$\frac{\omega^2}{c^2} z_{yy} - \left(1 - \frac{\omega^2}{c^2}\right) z_y - z_{\eta\eta} = 0. \tag{15}$$

The solution of equations (15) and (14) gives a complete picture of the motion, determining $z(\eta, y)$ and $\tau(p, \eta)$ for arbitrary $\psi(z)$ (with an arbitrarily prescribed entropy distribution), in particular for motions with shock waves.

In the nonrelativistic limit $w/mn \approx c^2 + W$, where W is the nonrelativistic thermal function per unit mass, and condition (10) has the form $V = f_1(p) + f_2(S)$, where V is the specific volume.

It is important to note that equation (15), derived above for the case $w = w(p)$ from the system (1)–(3), which contains the law of conservation of particles (3), can also be obtained from the laws of conservation of momentum and energy $\partial T_{ik}/\partial x_k = 0$, which for $w = w(p)$ form a closed system. Namely, projecting $\partial T_{ik}/\partial x_k = 0$ in the direction U_k and in the perpendicular direction, we obtain the system:

$$\frac{\partial}{\partial x_k}\left[\frac{wu_k}{B(p)}\right]=0,\qquad \frac{wu_k}{B(p)}\frac{\partial}{\partial x_k}\left[B(p)u_\alpha\right]=-\frac{\partial p}{\partial x_\alpha},\qquad B(p)=\exp\int\frac{dp}{w(p)}. \tag{16}$$

Introducing z by the formulas $wu_1/B(p)=-\partial z/\partial\tau$, $wu_4/B(p)=i\,\partial z/\partial x_1$ and carrying out transformations analogous to the preceding ones, for $z(\eta,y)$, where $y=\ln B(p)=\int[w(p)]^{-1}\,dp$, we obtain equation (15).

For an ultrarelativistic gas ($w=4p,\ 3\omega^2=c^2$) this equation was obtained by K. P. Stanyukovich[6]. It is analogous to that considered by I. M. Khalatnikov[4] and is of the type of the "telegraph" equation[7].

When considering motions of an ultrarelativistic and also photon gas, when the number of particles is not conserved, it is convenient to introduce an auxiliary gas by means of (16). In this case it is barotropic with velocity $\varphi_k=2k^{-1/2}p^{1/4}u_k$, pressure p, and density $\tilde{\rho}=kp^{1/2}$, where k is a constant.

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Note: Figure translations are in progress. See original paper for figures.