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A. S. MARKUS

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Abstract

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A. S. MARKUS

ON EXPANSION IN ROOT VECTORS OF A WEAKLY PERTURBED SELF-ADJOINT OPERATOR

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Let B be a linear operator acting in a separable Hilbert space $\mathfrak{H}$; let H be a self-adjoint operator acting in $\mathfrak{H}$ with discrete spectrum* and let $\{\mu_j\}_1^\infty$ be the sequence of all its distinct eigenvalues, arranged in increasing order of modulus. For simplicity we shall assume that $\mu_1 \neq 0$** and, consequently, that there exists an operator H^{-1}, which is completely continuous.

If the operator BH^{-1} is completely continuous and, for some $p > 0$,

$$\sum_{j=1}^{\infty} v_j |\mu_j|^{-p} < \infty, \tag{1}$$

where v_j is the multiplicity of the eigenvalue μ_j, then, by virtue of a well-known result of M. V. Keldysh (1), the operator $A = H + B$ has a discrete spectrum $\{\lambda_j\}_1^\infty$, and the system of its root subspaces $\{\mathfrak{S}_A(\lambda_j)\}_1^\infty$ is complete in $\mathfrak{H}$. In the present note it is established that, under certain additional restrictions on the operator B (depending on the convergence exponent p), one can choose a Bari basis (for the definition see (2,3)) of the space $\mathfrak{H}$, composed of subspaces, each of which is the direct sum of a finite number of root subspaces of the operator A. Consequently, there exists an increasing sequence of natural numbers $\{m_j\}_0^\infty$ ($m_0 = 1$) such that every vector $f \in \mathfrak{H}$ expands in a series "with parentheses" with respect to the system $\{\varphi_k\}_1^\infty$ of root vectors of the operator A:

$$f = \sum_{j=1}^{\infty} \left(\sum_{k=m_{j-1}}^{m_j - 1} \alpha_k \varphi_k \right). \tag{2}$$

We note that some conditions for the existence of a basis of eigenvectors and root vectors, and of root subspaces, of a dissipative operator were indicated by B. R. Mukminov (4), I. M. Glazman (5), and the author (3). In the work of V. B. Lidskii (6), broad conditions for the summability of the series (2) by a certain Abel method were obtained.

1. Theorem 1. *Let $0 < p < 1$, and suppose that the operators H and B satisfy at least one of the following two conditions:*

a) *the operator BH^{p-1} is bounded** and*

$$\lim_{k\to\infty} k|\mu_k|^{-p} = 0; \tag{3}$$

b) the operator BH^{p-1} is completely continuous and

$$\lim_{k\to\infty} k|\mu_k|^{-p} < \infty. \tag{4}$$

Then the distinct eigenvalues of the operator $A = H + B$ can be arranged in a sequence $\{\lambda_n\}_1^\infty$ such that, for some increasing sequence of natural numbers $\{n_j\}_1^\infty$ $(n_0 = 1)$, the sequence of subspaces $\{\mathfrak{N}_j\}_1^\infty$, where $\mathfrak{N}_j = \mathfrak{E}_A(\lambda_{n_{j-1}}) + \dots + \mathfrak{E}_A(\lambda_{n_j-1})$ $(j = 1, 2, \dots)$, is a Bari basis of the space $\mathfrak{H}$.

We indicate the main points of the proof of Theorem 1. Denote the sequences of positive and negative eigenvalues of the operator H by $\{\mu_k^+\}_1^\infty$ and $\{\mu_k^-\}_1^\infty$ $(\mu_k^+ < \mu_{k+1}^+,\ \mu_k^- > \mu_{k+1}^-;\ k = 1, 2, \dots)$, and put $\Delta\mu_k^+ = \mu_{k+1}^+ - \mu_k^+$, $\Delta\mu_k^- = \mu_{k+1}^- - \mu_k^-$. One can choose increasing sequences of natural numbers $\{t_j\}_1^\infty$ and $\{l_j\}_1^\infty$ such that

$$\inf_j \Delta\mu_{t_j}^+ (\mu_{t_j}^+)^{p-1} > 0, \qquad \inf_j |\Delta\mu_{l_j}^-|\,|\mu_{l_j}^-|^{p-1} > 0, \tag{5}$$

and if relation (3) is satisfied, then, moreover,

$$\lim_{j\to\infty} \Delta\mu_{t_j}^+ (\mu_{t_j}^+)^{p-1} = \infty, \qquad \lim_{j\to\infty} |\Delta\mu_{l_j}^-|\,|\mu_{l_j}^-|^{p-1} = \infty. \tag{6}$$

Put $\rho_k^+ = \frac{1}{2}(\mu_{t_k}^+ + \mu_{t_k+1}^+)$, $\rho_k^- = \frac{1}{2}|\mu_{l_k}^- + \mu_{l_k+1}^-|$, and denote by Γ_k the closed contour consisting of the semicircle $\Gamma_k^+ = \{\lambda : |\lambda| = \rho_k^+,\ \operatorname{Re}\lambda \geqslant 0\}$, the semicircle $\Gamma_k^- = \{\lambda : |\lambda| = \rho_k^-,\ \operatorname{Re}\lambda \leqslant 0\}$, and two segments of the imaginary axis: the

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* A linear operator is called an operator with discrete spectrum if its entire spectrum consists of eigenvalues of finite multiplicity with the only limit point at infinity.

** This restriction is inessential for what follows, since if it is not satisfied, one may replace the operator H by the operator $H - \lambda I$, where λ is any real regular value of the operator H.

*** For self-adjoint H and $\beta > 0$ we put

$$H^\beta = H_+^\beta + e^{i\beta\pi} H_-^\beta \quad \text{and} \quad H^{-\beta} = (H^{-1})^\beta.$$

segment γ_k^+ with endpoints $i\rho_k^+$ and $i\rho_k^-$, and the segment γ_k^- with endpoints $-i\rho_k^+$ and $-i\rho_k^-$. Let $R_\lambda = (H - \lambda I)^{-1}$ and $u_k(\lambda) = (\mu_{t_k}^+)^{1-p}|\lambda - \mu_{t_k}^+|^{-1}$. It can be shown that

$$|BR_\lambda| \leqslant c u_k(\lambda) \qquad (\lambda \in \Gamma_k^+;\ k = 1, 2, \ldots), \tag{7}$$

where c depends only on p and $|BH^{p-1}|$; and if condition b) is satisfied, then, moreover,

$$\lim_{k\to\infty} \max_{\lambda\in\Gamma_k^+} \frac{|BR_\lambda|}{u_k(\lambda)} = 0. \tag{8}$$

Similarly, $|BR_\lambda|$ is estimated for $\lambda \in \Gamma_k^-$. For points λ of the segments γ_k^+ and γ_k^- the estimate

$$|BR_\lambda| \leqslant c_1 |\lambda|^{-p}, \tag{9}$$

holds, where c_1 depends only on p and $|BH^{p-1}|$.

From the estimates given above and relations (5), (6) it follows that

$$\lim_{k\to\infty} \max_{\lambda\in\Gamma_k} |BR_\lambda| = 0.$$

Without loss of generality one may assume that $|BR_\lambda| \leqslant q < 1$ $(\lambda \in \Gamma_k;\ k = 1, 2, \ldots)$. Then all points $\lambda \in \Gamma_k$ $(k = 1, 2, \ldots)$ are regular points of the operator A, and

$$|\widetilde{R}_\lambda| \leqslant (1 - q)^{-1} |R_\lambda| \qquad (\widetilde{R}_\lambda = (A - \lambda I)^{-1},\ \lambda \in \Gamma_k;\ k = 1, 2, \ldots). \tag{10}$$

Put

$$\widetilde{P}_k = -\frac{1}{2\pi i} \int_{\Gamma_k} \widetilde{R}_\lambda \, d\lambda, \qquad P_k = -\frac{1}{2\pi i} \int_{\Gamma_k} R_\lambda \, d\lambda.$$

Since $R_\lambda - \widetilde{R}_\lambda = \widetilde{R}_\lambda B R_\lambda$, it follows, by virtue of (10), that

$$|\widetilde{P}_k - P_k| \leq [2\pi(1 - q)]^{-1} \int_{\Gamma_k} |R_\lambda| \, |BR_\lambda| \, |d\lambda|.$$

Using the estimates (7), (8), and (9) for $|BR_\lambda|$, the usual estimate for the norm of the resolvent R_λ of the self-adjoint operator H, and relations (5), (6), it is not difficult to show that

$$\lim_{k\to\infty} |\widetilde{P}_k - P_k| = 0.$$

Without loss of generality one may assume that

$$\sum_{k=1}^{\infty} |\widetilde{P}_k - P_k|^2 < \frac{1}{2}. \tag{11}$$

Denote by Q_k the orthogonal projector onto the orthogonal sum of the eigenspaces of the operator H corresponding to eigenvalues lying between Γ_{k-1} and Γ_k, and by $\widetilde{Q}_k$ the orthogonal projector onto the direct sum $\mathfrak{N}_k$ of the root subspaces of the operator A corresponding to eigenvalues lying between Γ_{k-1} and Γ_k.

From inequality (11) it follows easily that

$$\sum_{k=1}^{\infty} |\widetilde{Q}_k - Q_k|^2 < 1.$$

Thus, the sequence $\{\mathfrak{N}_k\}_1^{\infty}$ is a Bari basis of the space $\mathfrak{H}$ (see ([3]), Theorem 1).

Remark 1. Condition (3) is satisfied if $\sum_{k=1}^{\infty} |\mu_k|^{-p} < \infty$ (and, in particular, if (1) holds). We note that in the formulation of Theorem 1 condition (3) may be replaced by the more general conditions

$$\varlimsup_{k\to\infty} \Delta\mu_k^+ (\mu_k^+)^{p-1} = \infty, \qquad \varlimsup_{k\to\infty} |\Delta\mu_k^-|\, |\mu_k^-|^{p-1} = \infty,$$

and condition (4) by the conditions

$$\varliminf_{k\to\infty} \Delta\mu_k^+ (\mu_k^+)^{p-1} > 0, \qquad \varliminf_{k\to\infty} |\Delta\mu_k^-|\, |\mu_k^-|^{p-1} > 0.$$

Remark 2. If the operator H is semibounded, then one may assume that the eigenvalues of the operator A are arranged in the sequence $\{\lambda_n\}_1^{\infty}$ in increasing order of modulus.

Remark 3. If all eigenvalues of the operator H, starting from some point, are simple, the operator BH^{p-1} is bounded and

$$\sum_{k=1}^{\infty} [\Delta\mu_k^+ (\mu_k^+)^{p-1}]^{-2} < \infty, \qquad \sum_{k=1}^{\infty} (|\Delta\mu_k^-|\, |\mu_k^-|^{p-1})^{-2} < \infty,$$

then the operator A also has all eigenvalues, starting from some point, simple, and the system of root vectors of the operator A (among which only a finite number are not eigenvectors) forms a Bari basis of the space $\mathfrak{H}$.

2. The conditions of Theorem 1 are satisfied if as H one chooses the operator generated in the space $L_2(a,b)$ by the differential expression $i^n y^{(n)}$ ($n \geq 2$) and by any self-adjoint boundary conditions, i.e. linearly independent conditions

$$\sum_{k=1}^{n} \alpha_{jk} y^{(k-1)}(a) + \sum_{k=1}^{n} \beta_{jk} y^{(k-1)}(b) = 0 \quad (j = 1, 2, \dots, n), \tag{12}$$

whose coefficients satisfy the equalities

$$\sum_{k=1}^{n} (-1)^k (\alpha_{jk} \overline{\alpha}_{m,n-k+1} - \beta_{jk} \overline{\beta}_{m,n-k+1}) = 0 \quad (j, m = 1, 2, \dots, n),$$

and as B the operator generated in $\mathcal{L}_2(a,b)$ by the differential expression

$$p_1(x) y^{(n-2)} + p_2(x) y^{(n-3)} + \dots + p_{n-1}(x) y, \tag{13}$$

where $p_k(x)$ ($k = 1, 2, \dots, n-1$) are arbitrary measurable essentially bounded complex-valued functions. If, moreover, the conditions (12) are decomposing,* then the operators H and B satisfy the conditions of Remark 3.**

Theorem 2. Let $p \geqslant 1$, and let the operator H and the bounded operator B satisfy at least one of the following two conditions:

a) the operator $H^{p-1}B$ is bounded and

$$\lim_{k \to \infty} k |\mu_k|^{-p} = 0;$$

b) the operator $H^{p-1}B$ is completely continuous and

$$\lim_{k \to \infty} k |\mu_k|^{-p} < \infty.$$

Then, for the operator $A = H + B$, the assertion of Theorem 1 is valid.

The proof is in the main analogous to the proof of Theorem 1.

It should be noted that the completeness of the system of root vectors of the operator $H + B$ under the condition that essentially coincides with condition a) of Theorem 2 for $p = 1$ was established by M. A. Naimark ((9), Remark 5).

With the aid of Theorems 1 and 2, by passing to inverse operators, one proves without difficulty:

Theorem 3. Let $p > 0$, let C be a self-adjoint completely continuous operator, and let $\{\mu_k\}_1^\infty$ ($|\mu_k| > |\mu_{k+1}|$, $k = 1, 2, \dots$) be the sequence of all its distinct eigenvalues, and let T be some completely continuous operator. If the operator $A = C(I + T)$ is annulled only at zero and if at least one of the following two conditions is fulfilled:

a) the operator TC^{-p} is bounded and

$$\lim_{k\to\infty} k|\mu_k|^p = 0;$$

b) the operator TC^{-p} is completely continuous and

$$\lim_{k\to\infty} k|\mu_k|^p < \infty,$$

then, for the operator A, the assertion of Theorem 1 is valid.

The remarks to Theorem 1 (with the corresponding changes in the formulations) remain in force also for Theorems 2 and 3.

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Institute of Physics and Mathematics
of the Moldavian Branch of the Academy of Sciences of the USSR

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* This is possible only in the case when n is even and $n/2$ conditions are assigned to each endpoint.

** The last result intersects with the results of Kramer [7], from which it follows that the system of root vectors of the operator generated in $\mathcal{L}_2(a,b)$ by the differential expression of even order

$$y^{(n)} + p_0(x)y^{(n-1)} + \cdots + p_{n-1}(x)y$$

and by decomposing boundary conditions such that $n/2$ conditions are assigned to each endpoint forms a Riesz basis in $\mathcal{L}_2(a,b)$ (for the definition see [8]). Let us note, however, that if one restricts oneself to Bari bases, then the conditions of Theorem 1 are in a certain sense sharp. Namely, the example of the operator A, generated in $\mathcal{L}_2(0,1)$ by the differential expression $y'' + y'$ and the boundary conditions $y(0) = y(1) = 0$, shows that Theorem 1, generally speaking, ceases to be true for the operator $A = H + B$ if in (13) the term $p_0(x)y^{(n-1)}$ is added. This same example shows that in the formulation of Theorem 1 one cannot replace conditions a), b) by the condition: the operator BH^{p-1} is bounded and

$$\lim_{k\to\infty} k|\mu_k|^{-p} < \infty.$$

Note: Figure translations are in progress. See original paper for figures.

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