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Abstract

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I. V. STANKEVICH

ON THE THEORY OF PERTURBATION OF THE CONTINUOUS SPECTRUM

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The stability of the absolutely continuous part of the spectrum of a self-adjoint operator H_0 in a Hilbert space $\mathcal{H}$ with respect to perturbation by a symmetric operator V was investigated in the works ([1–10]). Kuroda in ([10]) showed that this problem reduces to the study of families of operators of the form $U_s(H, H_0) = e^{isH}e^{-isH_0}P_0$, $U_s(H_0, H) = e^{isH_0}e^{-isH}P$, where $H = H_0 + V$* and P_0 (P) is the projector onto the absolutely continuous subspace $M_0 \in \mathcal{H}$ $(M \in \mathcal{H})$ with respect to the operator H_0 (H). He proved that from the existence of the operators

$$U_\pm(H, H_0) = \lim_{s\to\pm\infty} U_s(H, H_0), \qquad U_\pm(H_0, H) = \lim_{s\to\pm\infty} U_s(H_0, H)$$

(the limits being understood in the sense of strong convergence of sequences of operators in $\mathcal{H}$) there follows the unitary equivalence of the absolutely continuous parts of the operators H_0 and H:

$$HP = U_+(H, H_0)H_0P_0U_+(H_0, H)$$

$$(U_+(H, H_0)U_+(H_0, H) = P).$$

The operators $U_\pm(H, H_0)$ define the generalized scattering operator S,

$$S = U_+^*(H, H_0)U_-(H, H_0),$$

which plays an important role in physics.

As Kuroda showed in ([7]), the operators $U_\pm(H, H_0)$ and $U_\pm(H_0, H)$ exist if the symmetric operator V satisfies the conditions: 1) $\mathcal{D}(V) \supset \mathcal{D}(H_0)$, where $\mathcal{D}(A)$ denotes the domain of definition of the operator A; 2) for any element f, $f \in \mathcal{D}(H_0)$,

$$\|Vf\| \le a\|H_0f\| + b\|f\|,$$

where $0 \le a < 1$, $b > 0$; 3) $|V|^{1/2}(H_0 + i)^{-1} \in \mathfrak{S}_2$**.

However, Kuroda's result does not cover a number of important cases. Thus, for example, for the Schrödinger operator in the space $L_2(E_k)$ (E_k is k-dimensional real Euclidean space), even in the case of an arbitrarily smooth potential function $V(x)$, this result is applicable only for $k \le 3$.

In the present work we generalize Kuroda' s result in the following way.

Theorem 1. Let

$$H_0 = \int \lambda \, dE^0_\lambda$$

be a self-adjoint operator with domain of definition $\mathcal{D}(H_0)$, and let V be such a symmetric operator that: 1) $\mathcal{D}(V) \supset \mathcal{D}(H_0)$; 2) there exists an integer positive number n such that, for every integer l, $0 \le l \le n-1$, and f, $f \in \mathcal{D}(H_0^{l+1})$:

$$\text{a) } Vf \in \mathcal{D}(H_0^l); \qquad \text{b) } \|H_0^{lVf}\| \le a\|H_0^{l+1} f\| + \sum_{j=0}^{l} b_{l,j} \|H_0^j f\|$$

and

$$0 \le a < \frac{1}{1 + 2^{2(n-1)}}, \qquad b_{l,j} \ge 0;$$

3) $|V|^{1/2}(H_0 + i)^{-n} \in \mathfrak{S}_2$.

Then the operators $U_\pm(H, H_0)$ and $U_\pm(H_0, H)$ exist***, where $H = H_0 + V$.

In particular, for $n = 1$ Kuroda' s theorem follows from our result****. In addition, our result is applicable to the multidimensional Schrödinger operator, for example, with a potential function $V(x)$ satisfying the conditions: 1) $V(x) \in L_1(E_k)$; 2) $V(x)$ has bounded derivatives up to order $2m$, where m satisfies the condition $4m - k > 0$. In this case

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* The operator $H = H_0 + V$ is assumed to be self-adjoint.

** $\mathfrak{S}_2$ denotes the class of Hilbert–Schmidt operators.

*** From conditions 1) and 2b) of Theorem 1 there follows, as was shown, for example, by Kato in ([9]), the self-adjointness of the operator H on $\mathcal{D}(H_0)$.

**** From our result Kuroda's theorem follows for $0 \le a < 1/2$. But, as Kuroda showed in ([7]), this is sufficient for the validity of the theorem for any a in $[0, 1)$.

the operator H_0 is the operator $-\Delta$, where Δ is the Laplace operator in $L_2(E_k)$, and the operator V is the operator of multiplication by the real function $V(x)$.

Let us verify that the conditions of Theorem 1 are satisfied for n satisfying the inequalities $[k/4] < n \leqslant m$. Indeed, if $f \in \mathcal{D}(H_0^{l+1})$, $0 \leqslant l \leqslant n-1$, l an integer, then the function $g = Vf$ has all derivatives of order $2l$ from $L_2(E_k)$, and therefore

$$g = \int_{E_k} \frac{e^{i(p,x)} \varphi(p)}{(p^2 + i)^l} \, d^k p \quad (\varphi(p) \in L_2(E_k)),$$

and, consequently, $g \in D(H_0^l)$. Moreover, the kernel of the operator $|V|^{1/2}(H_0 + i)^{-n}$ is equal to

$$K(x, y) = |V(x)|^{1/2} \int_{E_k} \frac{e^{i(p, x-y)}}{(p^2 + i)^n} \, d^k p,$$

therefore

$$\int |K(x,y)|^2 \, d^k x \, d^k y < \infty$$

and

$$|V|^{1/2}(H_0 + i)^{-n} \in \mathfrak{S}_2.$$

The fulfillment of conditions 1) and 2b) of Theorem 1 is obvious. For the proof of Theorem 1 we shall need several auxiliary theorems.

Theorem 2*. *Let H_0 and V be the same as in Theorem 1. Then: 1) the operator $H = H_0 + V$ on the set $\mathcal{D}(H_0)$ is self-adjoint; 2) $\mathcal{D}(H_0^n) = \mathcal{D}(H^n)$.*

Proof. It is enough to prove that $\mathcal{D}(H_0^n) = \mathcal{D}(H^n)$, since the first assertion of the theorem was proved by Kato in ([9]).

Let us show that $\mathcal{D}(H_0^n) \supset \mathcal{D}(H^n)$. Let $f \in \mathcal{D}(H^n)$. Then $Hf \in \mathcal{D}(H_0^{n-1})$. Suppose that we have proved for some integer l, $0 \leqslant l < n$, that $H^l f \in \mathcal{D}(H_0^{n-l})$. Let us show that $H^{l+1} f \in \mathcal{D}(H_0^{n-l-1})$. Indeed,

$$H^{l+1} f = (H_0 + V) H^l f = H_0 H^l f + V H^l f,$$

and, by condition 2a) of Theorem 1, $H^{l+1} f \in \mathcal{D}(H_0^{n-l-1})$. It follows that $f \in \mathcal{D}(H^n)$ and $\mathcal{D}(H_0^n) \supset \mathcal{D}(H^n)$.

Let us prove that the operator H^n on the set $\mathcal{D}(H_0^n)$ is self-adjoint. For this, we shall show that, for sufficiently large k, $k > 0$, the operators $H^n \pm ik$ map $\mathcal{D}(H_0^n)$ onto all of $\mathcal{H}$. From condition 2b) of Theorem 1 it follows that, for sufficiently large k,

$$\left\| T(H_0^n \pm ik)^{-1} \right\| < 1,$$

where $Tf = (H^n - H_0^n) f$ for $f \in \mathcal{D}(H_0^n)$. Therefore the operator

$$E + T(H_0^n \pm ik)^{-1},$$

where E is the identity operator, maps $\mathcal{H}$ onto all of $\mathcal{H}$. From the equality

$$H^n \pm ik = H_0^n + T \pm ik = (1 + T(H_0^n \pm ik)^{-1})(H_0^n \pm ik)$$

it follows that, for sufficiently large k, the range of the operator $H^n \pm ik$ coincides with the whole space. Therefore H^n on the set $\mathcal{D}(H_0^n)$ is self-adjoint and $\mathcal{D}(H_0^n) = \mathcal{D}(H^n)$. Theorem 2 is proved.

Theorem 3. *Let H_0 and V be the same as in Theorem 1. Then there exist constants a_l' and b_{lj}, $0 \leqslant a_l' < 1$, $b_{lj} \geqslant 0$, $l = 0, \dots, n-1$, $j = 0, \dots, l$, such that for any f, $f \in \mathcal{D}(H^{l+1})$,*

$$\|H^l V f\| \leqslant a_l' \|H^{l+1} f\| + \sum_{i=0}^{k} b_{li} \|H^i f\|. \tag{1}$$

Let us outline the proof. Inequality (1) for $l = 1$ was proved by Kuroda in ([7]). Suppose that (1) is true for $l = 0, 1, \dots, j-2$, where $j < n+1$. Using condition 2b) of Theorem 1, it is not difficult to show that then (1) is also fulfilled for $l = j-1$.

Theorem 4.** *Let the operators H_0 and V satisfy the conditions of Theorem 1 and, in addition, let V be self-adjoint and $0 \leqslant a < 1$. Then there exists a sequence $\{V_k\}$ of self-adjoint operators possessing the following properties:*

1) $V_k \in \mathfrak{S}_1$***;
2) $\left\| |V_k|^{1/2}(H_0+i)^{-n} \right\|_2 < c < \infty$ (c does not depend on k);

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* Theorem 2 is a generalization of Kato' s result ([9]) for the case $n = 1$.

** This theorem is a generalization of Kuroda' s analogous theorem for $n = 1$.

*** This means that $|V_k|^{1/2} \in \mathfrak{S}_2$.

3) $\|(H_0+i)^n(H_k+i)^{-n}\| < M < \infty$, $H_k = H_0 + V_k$, M does not depend on k;
4) $\lim\limits_{k\to\infty} e^{itH_k} = e^{itH}$ (in the strong sense);
5)
$$\lim_{k\to\infty}\int_s^\infty \left\| |V_k|^{1/2} e^{-itH_0} f \right\|^2 dt = \int_s^\infty \left\| |V|^{1/2} e^{-itH_0} f \right\|^2 dt$$
for any f, $f \in \mathcal{L}$, where $\mathcal{L}$ denotes the set of elements f, $f \in \mathcal{D}(H_0 + i)^n \cap M_0$, such that
$$d\left\|E^0_\lambda (H_0+i)^n f\right\|^2 / d\lambda \leqslant m^2, \quad (E^0_l - E^0_{-l})f = f$$
for some positive numbers m^2 and l.

Proof. Put
$$A_k = |V|^{1/2}(1+ik^{-1}H_0)^{-n}$$
and
$$B_k = (1-ik^{-1}H_0)^{-n}|V|^{1/2}W,$$
where $V = |V|^{1/2}W|V|^{1/2}$ and $\|W\| = 1$. Then define the operator $V_k = B_{kA}k$. Since B_k and A_k are of $\mathfrak{S}_2$, it follows that $V_k \in \mathfrak{S}_1$,
$$V_k = (1-ik^{-1}H_0)^{-n}V(1+ik^{-1}H_0)^{-n}.$$
On the set $\mathcal{D}(H_0)$ define the operator $H_k = H_0 + V_k$. Since V_k is a bounded operator, H_k is self-adjoint on $\mathcal{D}(H_0)$. Since
$$|V_k| = (1-ik^{-1}H_0)^{-n}|V|(1+ik^{-1}H_0)^{-n},$$

we have

$$\left\| |V_k|^{1/2} f \right\|^2 = (|V_k| f, f) = \|A_{kf}\|^2,$$

and, consequently,

$$\left\| |V_k|^{1/2}(H_0 + i)^{-n} \right\|_2 = \|A_k (H_0 + i)^{-n}\|_2 \leqslant \left\| |V|^{1/2}(H_0 + i)^{-n} \right\|_2 = c < \infty.$$

Assertion 3) is proved with the aid of Theorem 2 and inequalities 2b) of Theorem 1.

Assertions 4) and 5) are proved analogously to the case $n = 1$, which was considered by Kuroda in (7).

Theorem 5. Let the operators H_0 and V be the same as in Theorem 1 and, in addition, let V be self-adjoint. Then, if $U_\pm(H, H_0)$ exists, then for any $f, \ f \in \mathcal{L}$ (see Theorem 4), the inequality

$$\|(U_t - U_s) f\|^2 \leqslant c\,[\eta(t, f) + \eta(s, f)] \tag{2}$$

holds, where c and η have the form

$$c = \left[8m^2\pi \left\| |V|^{1/2}(H_0 + i)^{-n} \right\|_2^2 \|(H_0 + i)^n (H + i)^{-n}\|^2 \right]^{1/4},$$

$$\eta(t, f) = \left[\int_t^\infty \left\| |V|^{1/2} e^{-itH_0} f \right\|^2 dt \right]^{1/4}.$$

Conversely, if for any $f, \ f \in \mathcal{L}$, inequality (2) holds with a constant c independent of t and s, then the operator $U_\pm(H, H_0)$ exists.

Proof. Suppose, for definiteness, that the operator $U_+(H, H_0)$ exists. Then

$$e^{-itH} U_+ = U_+ e^{-itH_0},$$

and therefore

$$\|(U_s - U_t) f\|^2 = 2 \text{ Re } i \int_s^\infty (V e^{-itH_0}, \, U_+ e^{itH_0} f) \, dt.$$

Using the representation V in the form

$$V = |V|^{1/2} W |V|^{1/2},$$

where $\|W\| = 1$, and then estimating the integral with the aid of Schwarz' s inequality, we obtain

$$\|(U_+ - U_s) f\|^2 \leqslant 2 \left[\int_s^\infty \left\| |V|^{1/2} e^{-itH_0} f \right\|^2 dt \right]^{1/2} \left[\int_s^\infty \left\| |V|^{1/2} U_+ e^{-itH_0} f \right\|^2 dt \right]^{1/2}.$$

Using the equality $U_+ H_0 P_0 = H U_+$, we transform the integrand in the second integral to the form

$$\left\| |V|^{1/2} (H_0 + i)^{-n} (H_0 + i)^n (H + i)^{-n} U_+ e^{-itH_0} (H_0 + i)^n f \right\|^2.$$

Using results, for example, of Kuroda (7), it is not hard to show that

$$\|(U_+ - U_s) f\| \leqslant \left[8m^2 \pi \left\| |V|^{1/2} (H_0 + i)^{-n} \right\|_2 \|(H_0 + i)^n (H + i)^{-n}\| \right]^{1/4} \eta(s, f).$$

From this the direct assertion of the lemma follows.

Conversely, suppose that (2) is true for any $f \in \mathcal{L}$. Since the integrals $\eta(t, f)$ and $\eta(s, f)$ converge, as $t, s \to \infty$ the right-hand side of (2) tends to zero. But in (7) it is proved that $\mathcal{L}$ is dense in M_0. Moreover, $\|U_t\| = 1$. Therefore $U_+(H, H_0)$ exists. Theorem 5 is proved.

We pass to the proof of Theorem 1. Just as in the case $n = 1$, Theorem 1 reduces to an analogous theorem under the additional assumption that the operator V is self-adjoint. We shall prove Theorem 1 in this case.

From Kuroda' s result ([7]) it follows that for any natural number k, $k > 0$, there exists an operator $U_\pm(H_k, H_0)$, where $H_k = H_0 + V_k$ and

$$V_k = (1 - ik^{-1} H_0)^{-n} V (1 + ik^{-1} H_0)^{-n}, \quad V_k \in \mathfrak{S}_1.$$

Therefore, from Theorem 5, for any $f \in \mathfrak{L}$ inequality (2) follows for $U_t^{(k)}$, $U_s^{(k)}$, c_k, and η_k, which are defined in Theorem 5 ($U_t^{(k)} = e^{itH_k} e^{-itH_0} P_0$). Hence, and from Theorem 4, inequality (2) follows for H_0 and H with a constant c independent of t and s. By virtue of the converse assertion of Theorem 5, this proves the existence of the operator.

From Theorems 2, 3, and 4 it follows that if the operator H is taken as the unperturbed operator, then the operators H and $-V$ satisfy the conditions of Theorem 1 with constants $0 \leq a < 1$, $b_{lj} \geq 0$. This is sufficient for the existence of the operator $U_\pm(H_0, H)$. Theorem 1 is proved.

In conclusion let us consider an example. Let $\mathcal{H} = L_2(E_k)$; $H_0 = (-1)^t \Delta^t$ (t is a positive integer), where Δ is the Laplace operator in $L_2(E_k)$, and

$$\mathfrak{D}(H_0) = \left\{ \psi(x),\ \psi(x) = \int \frac{\varphi(p) e^{i(p,x)}}{p^{2t} + i} \, d^{kp},\ \varphi(p) \in L_2(E_k) \right\}.$$

On $\mathfrak{D}(H_0)$ define a symmetric differential operator of order α, where $0 \leq \alpha < 2t$:

$$V\psi(x) = \sum_{\substack{j_1 \cdots j_\beta \\ \beta \leq \alpha}} a_{j_1 \cdots j_\beta}(x) \frac{\partial^\beta}{\partial x_{j_1} \ldots \partial x_{j_\beta}} \psi(x)$$

and

$$(1 + |x|^4) a_{j_1 \cdots j_\beta}(x) \in L_1(E_k)$$

and $a_{j_1 \dots j_\beta}(x)$ have bounded derivatives up to order $2mt$, where m is a positive integer satisfying the condition $2tm - \alpha - k > 0$. The operators H_0 and V thus defined satisfy the conditions of Theorem 1 for any integer n,

$$\left[\frac{\alpha + k}{2t}\right] < n \leq m.$$

Indeed, the operator V maps any element f, $f \in \mathfrak{D}(H_0^l)$, $0 \leq l \leq n$, into an element g, $g = Vf$, and g has all derivatives up to order $2t(l-1)$ inclusive belonging to $L_2(E_k)$; therefore

$$g = \int \frac{\varphi(p) e^{i(p,x)}}{(p^2 + 1)^{t(l-1)}}\, d^{kp},$$

where $\varphi(p) \in L_2(E_k)$, and $g \in \mathfrak{D}(H_0^{l-1})$. Moreover, the operator

$$A = (H_0 - i)^{-n} |V| (H_0 + i)^{-n} = |(H_0 - i)^{-n} V (H_0 + i)^{-n}|,$$

and the kernel of the operator $(H_0 - i)^{-n} V (H_0 + i)^{-n}$ can be computed explicitly. Using results of A. O. Gelfond ([11]), it is not hard to show that

$$(H_0 - i)^{-n} V (H_0 + i)^{-n} \in \mathfrak{S}_1.$$

Therefore also the operator $A \in \mathfrak{S}_1$ and

$$|V|^{1/2} (H_0 + i)^{-n} \in \mathfrak{S}_2.$$

Condition 2b) of Theorem 1 is directly easy to verify.

Thus, the absolutely continuous part of the operator $(-\Delta)^t + V$, where V is a differential operator of order less than $2t$, with coefficients satisfying the conditions listed above, is unitarily equivalent to the operator $(-\Delta)^t$ in $L_2(E_k)$.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.