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Abstract

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MATHEMATICS

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A CONSTRUCTION PROBLEM CONNECTED WITH THE DISTRIBUTION OF THE FRACTIONAL PARTS OF AN EXPONENTIAL FUNCTION

(Presented by Academician I. M. Vinogradov, 17 XI 1961)

Let $\lambda > 1$ be a fixed number. It is required to construct a real number θ such that $\{\theta\lambda^x\}$, $x = 1, 2, \ldots$, are uniformly distributed. Many solutions of this problem are known in the case when $\lambda = g$, $g \geqslant 2$, is a given natural number. If λ is not equal to a natural number, then the problem has been investigated only for the case when λ is a Pisot number ([1]). In the present paper the problem is solved for the general case. It should be noted that the construction which we carry out is much more complicated than the constructions mentioned, although it is theoretically executable. We rely on Koksma' s theorem ([2]), which states that for almost all numbers θ (in the sense of Lebesgue measure) from a finite interval $(\alpha\beta)$ the fractional parts $\{\theta\lambda^x\}$ are uniformly distributed, and on Lebesgue' s principle ([3]), which makes it possible to render metric theorems effective. We shall construct the number θ from the interval $\{0, 1\}$.

Let $h \neq 0$ be an integer and ν a natural number. Denote by $E_{(h,\nu)}$ the set of those θ from the interval $0 \leqslant \theta \leqslant 1$ for which

$$\left| \frac{1}{\nu^2} \sum_{x=1}^{\nu^2} e^{2\pi i h\theta\lambda^x} \right| > \frac{1}{4\sqrt{\nu+1}}.$$

The following lemmas are easily proved:

Lemma 1. *The set $E_{(h,\nu)}$ consists of a finite number of intervals. The coordinates of the endpoints of these intervals can be computed.*

Denote these intervals by

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$$E_1^{(h,\nu)},\ E_2^{(h,\nu)},\ \dots,\ E_{l(h,\nu)}^{(h,\nu)}; \qquad E_{(h,\nu)} = \bigcup_{j=1}^{l(h,\nu)} E_j^{(h,\nu)}.$$

Lemma 2. *The inequality holds*

$$\operatorname{mes} E_{(h,\nu)} < \sqrt{\nu+1}\left(\frac{1}{\nu^2} + \frac{1}{\lambda-1}\frac{\log 3\nu^2}{\nu^2}\right).$$

Since the series

$$\sum_{\nu=1}^{\infty} \sqrt{\nu+1}\left(\frac{1}{\nu^2} + \frac{1}{\lambda-1}\frac{\log 3\nu^2}{\nu^2}\right)$$

converges, for a given integer $h \neq 0$ we can find $\nu_0(h)$ such that

$$\sum_{\nu=\nu_0(h)}^{\infty} \sqrt{\nu+1}\left(\frac{1}{\nu^2} + \frac{1}{\lambda-1}\frac{\log 3\nu^2}{\nu^2}\right) < \frac{1}{4^{|h|}}; \qquad \mathfrak{R}_\nu^{(h)} = \bigcup_{j=\nu}^{\infty} E_{(j,h)}.$$

Since $E_{(h,\nu)}$ is the sum of a finite number of intervals, the set $\mathfrak{R}_\nu^{(h)}$ consists of no more than a countable number of nonintersecting intervals. This, in particular, also applies to $\mathfrak{R}_{\nu_0(h)}^{(h)}$. For the measure of $\mathfrak{R}_{\nu_0(h)}^{(h)}$ we have the estimate

$$\operatorname{mes} \mathfrak{N}_{\nu_0(h)}^{(h)} \leq \sum_{\nu=\nu_0(h)}^{\infty} \sqrt{\nu+1}\left(\frac{1}{\nu^2} + \frac{1}{\lambda-1}\frac{\log 3\nu^2}{\nu^2}\right) < \frac{1}{4^{|h|}}.$$

Define the set $\mathfrak{M}$ in the following way:

$$\mathfrak{M} = \bigcup_{\substack{h=-\infty \\ h\neq 0}}^{\infty} \mathfrak{N}_{\nu_0(h)}^{(h)}.$$

The set $\mathfrak{M}$ consists of a countable number of intervals. Denote them by $E_1, E_2, \dots$. For the measure of $\mathfrak{M}$ the estimate

$$\operatorname{mes} \mathfrak{M} \leq 2\sum_{h=1}^{\infty} \frac{1}{4^{(h)}} = \frac{2}{3} < 1 \quad \text{(strictly!)}.$$

holds.

Let α_j be the length of the interval E_j. Denote

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$$S = \sum_{j=1}^{\infty} \alpha_j; \qquad S_P = \sum_{j=1}^{P} \alpha_j; \qquad r_P = S - S_P.$$

Consider the following 10 segments:

$$[0,\ {}^{1}/_{10}],\ [{}^{1}/_{10},\ {}^{2}/_{10}],\ [{}^{2}/_{10},\ {}^{3}/_{10}],\ \dots,\ [{}^{9}/_{10},\ 1].$$

Let p_1 be some number. Take the first p_1 terms of the system $E_1, E_2, \dots, E_{p_1}$. Let $S_0^{(p_1)}, S_1^{(p_1)}, \dots, S_9^{(p_1)}$ be the total lengths of the intersections of $E_1 \cup E_2 \cup \dots \cup E_{p_1}$, respectively, with $[0, {}^{1}/_{10}], [{}^{1}/_{10}, {}^{2}/_{10}], \dots, [{}^{9}/_{10}, 1]$. Then

$$S_0^{(p_1)} + S_1^{(p_1)} + \dots + S_9^{(p_1)} = S_{p_1}.$$

Hence

$$(S_0^{(p_1)} + r_{p_1}) + (S_1^{(p_1)} + r_{p_1}) + \dots + (S_9^{(p_1)} + r_{p_1}) = S_{p_1} + 10 r_{p_1} = S + 9 r_{p_1}.$$

Let p_1 be chosen so large that $S + 9r_{p_1} < 1$. Then there exists a segment

$$\left[\frac{i}{10}, \frac{i+1}{10}\right]$$

for which

$$S_i^{(p_1)} + r_{p_1} \le \frac{S + 9r_{p_1}}{10} < \frac{1}{10} \quad \text{(strictly!)}.$$

Take for i_1 the smallest number $0 \le i \le 9$ such that this inequality is satisfied.

Divide the segment

$$\left[\frac{i_1}{10}, \frac{i_1+1}{10}\right]$$

into 10 segments

$$\left[\frac{i_1}{10}, \frac{i_1}{10} + \frac{1}{100}\right], \left[\frac{i_1}{10} + \frac{1}{100}, \frac{i_1}{10} + \frac{2}{100}\right], \dots, \left[\frac{i_1}{10} + \frac{9}{100}, \frac{i_1+1}{10}\right].$$

Choose a number $p_2 > p_1$ so large that the inequality

$$S + 9r_{p_1} + 90 r_{p_2} < 1 \quad \text{(strictly!)}$$

will hold.

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Let i be a number, $0 \leq i \leq 9$. Denote by $S_{i_1 i}^{(p_2)}$ the length of the intersection of $E_1 \cup E_2 \cup \cdots \cup E_{p_2}$ with

$$\left[\frac{i_1}{10} + \frac{i}{100}, \frac{i_1}{10} + \frac{i+1}{100}\right].$$

Then

$$\sum_{i=0}^{9}\left(S_{i_1 i}^{(p_2)} + r_{p_2}\right) = \sum_{i=0}^{9} S_{i_1 i}^{(p_2)} + 10r_{p_2} = S_{i_1}^{(p_2)} + 10r_{p_2} =$$

$$= S_{i_1}^{(p_1)} + r_{p_1} - r_{p_2} + 10r_{p_2} = S_{i_1}^{(p_1)} + r_{p_1} + 9r_{p_2} \leq \frac{1}{10}(S + 9r_{p_1}) + 9r_{p_2} =$$

$$= \frac{1}{10}\left(S + 9r_{p_1} + 90r_{p_2}\right) < 1 \quad \text{(strictly!)}.$$

There will be found such an index l_2 that

$$S_{i_1 i_2}^{(p_2)} + r_{p_2} \leqslant \frac{1}{100}\left(S + 9r_{p_1} + 90r_{p_2}\right) < \frac{1}{100} \quad \text{(strictly!)}.$$

Then we define an index p_3 such that

$$S + 9r_{p_1} + 90r_{p_2} + 900r_{p_3} < 1,$$

and an interval of length $\frac{1}{1000}$ such that

$$S_{i_1 i_2 i_3}^{(p_3)} < \frac{S + 9r_{p_1} + 90r_{p_2} + 900r_{p_3}}{1000} < \frac{1}{1000}.$$

This process can be continued if the series

$$S + 9r_{p_1} + 90r_{p_2} + 900r_{p_3} + \cdots$$

converges and its sum is less than 1.

This can be done as follows. For p_1 take the first number such that

$$r_{p_1} < \frac{1 - S}{2 \cdot 9};$$

for p_k take the first number such that

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$$r_{p_k} < \frac{1-S}{2^k \cdot 9 \cdot 10^{k-1}}.$$

Then

$$S + 9r_{p_1} + 90r_{p_2} + 900r_{p_3} + \cdots$$

$$< S + \frac{1-S}{1} + \frac{1-S}{2^2} + \cdots = S + (1-S) = 1.$$

Define the number θ_0 by the equality

$$\theta_0 = \frac{i_1}{10} + \frac{i_2}{100} + \frac{i_3}{1000} + \cdots$$

It is obvious that the number θ_0 does not belong to $\mathfrak{M}$.

It is easy to prove that for every integer $h \neq 0$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x=1}^{N} e^{2\pi i h \theta_0 \lambda^x} = 0.$$

Hence, by Weyl' s criterion ([4]), it follows that the fractional parts $\{\theta_0 \lambda^x\}$ are uniformly distributed.

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Note: Figure translations are in progress. See original paper for figures.

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