Mathematics

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Abstract

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An Existence Theorem for a Bounded Solution of Nonlinear Singular Integral Equations with Cauchy Kernel

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One of the authors of the present article investigated (^{1}) a nonlinear singular integral equation of the form

$$u(x) = \lambda \int_a^b \frac{k[x, s, u(s)]}{s-x}\, ds. \tag{1}$$

By applying Schauder' s topological principle, an existence theorem was proved for a solution of equation (1) in the space $H^k_{\alpha,\beta,\delta}$, whose elements satisfy the conditions:

$$|u(x)| \leq \frac{k}{|x-a|^\alpha |x-b|^\beta}, \qquad |u(x+\Delta x) - u(x)| \leq \frac{k|\Delta x|^\delta}{|x-a|^{\alpha+\delta}|b-x-\Delta x|^{\beta+\delta}},$$

where $k = \text{const}$, $0 < \alpha, \beta < 1$, $0 < \alpha+\delta, \beta+\delta < 1$, $0 < \delta < 1$.

When an additional restriction was imposed on the function $k(x, s, u)$, the uniqueness of this solution and the applicability of the method of successive approximations were proved. Then A. A. Babaev showed in (^{2}) that the conditions found by A. I. Guseinov, under which Schauder' s principle is applied to prove the existence of at least one solution of equation (1), ensure the uniqueness of this solution and the applicability of the method of successive approximations for the equation

$$u(x) = \lambda \int_a^b \frac{H(x,s) f[s, u(s)]}{s-x}\, ds. \tag{1'}$$

The results obtained in (^{1}) for equation (1) were transferred by V. Pogorzelski (^{3}) to equations of the form

$$u(t) = \lambda \int_a^b \frac{k[t, \tau, u(\tau)]}{t-\tau}\, d\tau,$$

where L consists of a finite number of smooth closed arcs having no common points.

In papers ($^{1-3}$), the function $k(x, s, u)$ grows with respect to u no faster than a linear function. The aim of the present article is to prove an existence and uniqueness theorem for a bounded solution of equation (1) in the Hölder space $H_{k,\delta}$ (even if $k(x, s, u)$ grows arbitrarily fast with respect to u). The elements of the space $H_{k,\delta}$ on the segment $[a, b]$ satisfy the conditions:

$$|u(x)| \leq k, \qquad |u(x+\Delta x) - u(x)| \leq k|\Delta x|^{\delta},$$

where $k = \text{const}$, $0 < \delta < 1$.

We shall first state, without proof, the following basic lemmas:

Lemma 1. If $f(x, s)$ is defined for $a \leqslant x, s \leqslant b$ and satisfies the conditions

$$f(a, a) = f(b, b) = 0; \tag{2}$$

$$|f(x+\Delta x, s+\Delta s) - f(x, s)| \leqslant B\,[|\Delta x|^{\delta_1} + |\Delta s|^{\delta}], \tag{3}$$

then the function

$$W(x) = \int_a^b \frac{f(x, s)}{s-x}\, ds \tag{4}$$

for all $x \in [a, b]$ and $0 \leqslant \Delta x \leqslant \min\left\{\frac{|x-a|}{4}, \frac{|x-b|}{4}\right\}$ satisfies the conditions

$$|W(x)| \leqslant LB, \qquad |W(x+\Delta x) - W(x)| \leqslant LB\,|\Delta x|^{\delta},$$

where L is a numerical constant independent of B.

Remark 1. If the function $f(x, s)$ satisfies condition (3), then, for the boundedness of the function (4), condition (2) is necessary and sufficient.

Lemma 2. If the function $k[x, s, u(s)]$ is defined for $a \leqslant x, s \leqslant b$ and satisfies the conditions:

$$k[a, a, u] = k[b, b, u] = 0,$$

$$|k[x+\Delta x, s+\Delta s, u(s+\Delta s)] - k[x, s, u(s)]| \leqslant$$

$$\leqslant A_1|\Delta x|^{\delta_1} + A_2|\Delta s|^{\delta} + A_3|u(s+\Delta s) - u(s)|, \tag{5}$$

where $0 < \delta < \delta_1 < 1$, $0 \leqslant \Delta x \leqslant \min\left\{\frac{|x-a|}{4}, \frac{|b-x|}{4}\right\}$, $u(x) \in H_{k,\delta}$, then for all $x \in [a,b]$ and for

$$|\lambda| < \frac{k}{D(A_1 + A_2 + A_3 k)}$$

the operator

$$Bu = \lambda \int_a^b \frac{k[x, s, u(s)]}{s-x}\, ds$$

maps the space $H_{k,\delta}$ into itself, where D is a constant independent of $A_1 + A_2 + A_3 k$.

Lemma 3. If on $[a,b]$ the function $k[x,s,u(s)]$ satisfies the condition

$$|k[x+\Delta x, s+\Delta s, u(s+\Delta s)] - k[x, s, u(s)]| \leqslant$$

$$\leqslant A_1'|\Delta x|^{\delta_1} + A_2'|\Delta x|^{\delta} + A_3'|f(x,s)| \cdot |u(s+\Delta s) - u(s)| \tag{6}$$

and

$$u(x) \in H_{k,\delta},$$

where $0 < \delta < \delta_1 < 1$, and $f(x,s)$ satisfies the conditions of Lemma 1, then the operator Bu is continuous in the sense of the metric of the space of continuous functions for

$$0 \leqslant \Delta x \leqslant \min\left\{\frac{|x-a|}{4}, \frac{|b-x|}{4}\right\}.$$

Thus, by means of a continuous operator, the closed convex set $H_{k,\delta}$ of the space of continuous functions C is mapped into its compact part. Consequently, on the basis of the generalized Schauder principle there exists a fixed point, i.e. a solution of equation (1), and the following holds:

Theorem 1. If on $[a, b]$ the function $k[x, s, u(s)]$ satisfies conditions (5), (6), then there exists a number

$$\lambda_0 = \frac{k}{D\,[A_1' + A_2' + 2A_3' B(b-a)^{\delta_1} k]},$$

such that for $|\lambda| < \lambda_0$ the linear singular integral equation (1) has at least one solution $u(x) \in H_{k,\delta}$.

Now let us show the applicability of the method of successive approximations under one additional condition.

Let the function

$$g(x, s, u) = k(x, s, u) - k(s, s, u) \tag{*}$$

satisfy the condition

$$|g(x, s, u) - g(x, s, v)| \leq C|x - s|^{\alpha_1} |u - v|, \tag{**}$$

for all $a \leq x, s \leq b,\ -\infty < u < +\infty,\ 0 < \alpha_1 < 1$. Then the operator Bu maps the space $L_p(\rho)$ into $L_p(\rho)$ and satisfies the Lipschitz condition:

$$\|Bu - Bv\|_{L_p(\rho)} \leq |\lambda|\,[2A_3' B(b-a)^{\delta_1} F + CR_1]\,\|u - v\|_{L_p(\rho)},$$

where

$$1 < q < q_0, \qquad \frac{1}{p} + \frac{1}{q} = 1, \qquad 0 < q_0(1 - \alpha_1) + \alpha < 1,$$

$$0 < \alpha < \alpha_1 < 1, \qquad \rho(x) = (x-a)^{\alpha(p-1)}(b-x)^{2(p-1)},$$

$$R_1 = \left\{ \int_a^b \rho(x) \left[\int_a^b \rho^{-q/p}(s) |s - x|^{q(\alpha_1 - 1)}\, ds \right]^{p/q} dx \right\}^{1/p},$$

F is the norm of the linear singular operator with Cauchy kernel in $L_p(\rho)$. Since $H_{k,\delta}$ is complete in the metric of $L_p(\rho)$, the following is true.

Theorem 2. If $k[x, s, u]$ satisfies the conditions of Theorem 1 and condition $(**)$, then, for

$$|\lambda| < \min\left\{ \frac{k}{D[A_2' + A_2' + 2A_3'(b-a)^{\delta_1} Bk]}, \frac{1}{2BA_3'(b-a)^{\delta_1} F + CR_1} \right\},$$

equation (1) has a unique solution in the space $H_{k,\delta}$. The successive approximations will converge in the metric of the space $L_p(\rho)$.

Denote

$$H_\delta = \bigcup_{k>0} H_{k,\delta}.$$

$$\lim_{k\to\infty} \frac{k}{D[A_1' + A_2' + 2A_3(b-a)^{\delta_1} kB]} = \frac{1}{2BDA_3'(b-a)^{\delta_1}}.$$

Corollary. If the conditions of Theorem 2 are fulfilled and if

$$|\lambda| < \min\left\{\frac{1}{2DA_3'(b-a)^{\delta_1}B}, \frac{1}{2BA_3'(b-a)^{\delta_1}F + CR_1}\right\},$$

then equation (1) has a unique solution in H_δ; it can be found by the method of successive approximations, starting from any element of H_δ.

The successive approximations converge in the metric of $L_p(\rho)$.

In work (4) it was shown that convergence of elements from $H_{k,\delta}$ in the metric $L_p(\rho)$ and uniform convergence are equivalent.

Therefore these successive approximations will converge uniformly in the sense of the metric of the space of continuous functions, if as the zero approximation one takes a function from H_δ.

Remark 2. What was set forth above carries over to equations of the form

$$u(x) = \lambda \int_a^b \frac{f(x,s)u(s)}{s-x}\, ds + g(x),$$

if $f(x,s)$ satisfies the conditions of Lemma 1, $g(x) \in H_{k,\delta}$.

Remark 3. If the function $k[x,s,u]$ is represented in the form $H(x,s)f(s,u)$ and if $H(x,s)$ and $f(s,u)$ satisfy the conditions:

$$H(a,a) = H(b,b) = 0,$$

$$|H(x+\Delta x, s+\Delta s) - H(x,s)| \le B_1|\Delta x|^{\delta_1} + B_2|\Delta s|^{\delta},$$

$$|f(s+\Delta s, u+\Delta u) - f(s,u)| \le B_3|\Delta s|^{\delta} + B_4|\Delta u|,$$
$$0<\delta<\delta_1<1, \qquad a \le x,\ s \le b, \qquad -\infty < u < +\infty,$$

then condition $(**)$ is fulfilled automatically, and the conditions for the applicability of Schauder' s principle to equation (1) that we have found ensure uniqueness of the solution in H_δ and the applicability of the method of successive approximations to equation $(1')$.

Remark 4. Introduce in $H_{k,\delta}$ the metric

$$\rho_{L_p(\rho)}(u,v) = \left\{ \int_a^b \rho(x)|u(x)-v(x)|^p \, dx \right\}, \qquad p > 1.$$

As we have already said, $H_{k,\delta}$ is a complete metric space in the metric just introduced.

Let $k(x,s,u)$ satisfy the conditions of Theorem 1 and condition $(**)$ for all $a \le x,\ s \le b,\ -k \le u \le k < +\infty$. As was shown above, the operator Bu will be a contraction operator if

$$|\lambda| < \frac{1}{2BA_3'(b-a)^{\delta_1}F + CR_1},$$

where A_3', C, A_1', A_2' will be functions of k. If $k(x,s,u)$ grows faster than a linear function with respect to u, then, generally speaking,

$$\lim_{k\to\infty} \frac{1}{2BA_3'(b-a)^{\delta_1}F + CR_1} = 0.$$

Therefore, for fixed values of k, equation (1) will have a unique solution in $H_{k,\delta}$ (even if $k(x,s,u)$ grows arbitrarily fast with respect to u) for small values of the parameter λ. The successive approximations will converge in the sense of the metric of the space of continuous functions.

It is now clear that the equation

$$u(x) = \lambda \int_a^b \frac{f(x,s)P_n(u(s))}{s-x} \, ds, \tag{A}$$

where $P_n(u)$ is a polynomial of degree n with respect to u, and $f(x,s)$ satisfies the conditions of Lemma 1, has a unique solution in $H_{k,\delta}$, whose successive approximations converge uniformly in the sense of the metric of the space of continuous functions (for small values of λ).

Equations of the form (A) for $n > 1$ had not previously been successfully investigated.

By the same methods one can investigate the most general equation of the form

$$u(x) = \lambda F(x, W(x)),$$

where

$$W(x) = \int_a^b \frac{k[x, s, u(s)]}{s - x} \, ds.$$

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Note: Figure translations are in progress. See original paper for figures.