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Abstract

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ON THE REDUCTION OF CERTAIN THREE-DIMENSIONAL DIFFRACTION PROBLEMS TO THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION***

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1. Let us consider the solution $u(t, x, y, z)$ of the nonstationary problem of diffraction of an incident plane wave of the form $f(t-x)$ by a conical obstacle S, i.e., by an obstacle S that can be specified by the equation $F(x, y, z) = 0$, where F is a homogeneous function of x, y, z. For simplicity we shall assume that S is situated in the half-space $x > 0$. We have the following problem:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = 0; \tag{1}$$

$$u\big|_S = 0; \tag{2}$$

for $t < 0$

$$u(t, x, y, z) = f(t-x), \tag{3}$$

where $f(\xi) = 0$ for $\xi < 0$. Thus, the front of the incident wave has the equation $x = t$, and in front of the incident-wave front the solution is equal to zero. At $t = 0$ the incident-wave front touches S, and the vertex of S begins to emit a spherical wave whose front has the form $r = \sqrt{x^2 + y^2 + z^2} = t$. In many problems (below we shall give an example of such a problem) the solution for $r > t$, i.e., in front of the spherical-wave front, is determined comparatively simply. We shall show that, for a special choice of the form of the incident wave, in the case when the solution for $r > t$ is known, the problem of determining the solution for $r < t$ reduces to the Dirichlet problem for the Laplace equation. Namely, we take the incident wave in the form

$$f(t-x) = (t-x)_+^\lambda = \begin{cases} (t-x)^\lambda, & t > x, \\ 0, & t < x, \end{cases}$$

i.e., as a homogeneous function of dimension λ. Then, in view of the homogeneity of the boundary conditions, the solution will be a homogeneous function of the same dimension, and for $t > 0$, i.e., it will be representable in the form $t^\lambda v\left(\frac{r}{t}, \omega\right)$. Here $r = \sqrt{x^2 + y^2 + z^2}$, and ω is the direction of the vector x, y, z in three-dimensional space. Substituting the function $t^\lambda v(\xi, \omega)$, where $\xi = r/t$, into the wave equation (1), we obtain for $v(\xi, \omega)$ an equation which is elliptic for $\xi < 1$ and degenerates at $\xi = 1$. It turns out that for $\lambda = -1/2$ this equation can be reduced to the Laplace equation.

Theorem 1. Let the function $t^{-1/2} v\left(\frac{r}{t}, \omega\right)$ be a solution of equation (1). Then the function $v(\xi, \omega)$ can be represented in the form

$$v(\xi, \omega) = \left(1 + \sqrt{1 - \xi^2}\right)^{-1/2} w\left(\frac{\xi}{1 + \sqrt{1 - \xi^2}}, \omega\right),$$

where w is a harmonic function, i.e., $w(\rho, \omega)$ satisfies the Laplace equation (written in spherical coordinates):

$$\frac{\partial^2 w(\rho, \omega)}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial w(\rho, \omega)}{\partial \rho} + \frac{1}{\rho^2}\Delta_\omega w(\rho, \omega) = 0,$$

where Δ_ω is the angular part of the Laplace operator.

* Reported at the Odessa Symposium on Wave Diffraction (September 1960).

Therefore the problem of determining the solution of the wave equation inside the cone $r \leq t$, taking on the generators of the cone the values $u(t, t, \omega)$ and satisfying the boundary conditions (2), reduces, in the case where $u(t, t, \omega) = t^{-1/2} u_0(\omega)$, to the Dirichlet problem for the Laplace equation.

To this same problem, obviously, reduces the problem of nonstationary diffraction of an incident plane wave of the form $(t - x)_+^{-1/2}$ in the case where for $r > t$ the solution is known to us.

This fact allows us at once to indicate the type of singularity at the front of the wave scattered by the vertex S for any direction ω not lying on the boundary of shadow and light (if the direction ω lies on the boundary of shadow and light, then the boundary function for the Dirichlet problem has a singularity at the corresponding point). Namely, when one approaches from inside the sphere $r = t$, the solution itself remains continuous, while its derivative with respect to t tends to infinity as $c_1(\omega)/t\sqrt{t - r}$. The value of the coefficient $c_1(\omega)$ can be indicated after the numerical solution of the corresponding Dirichlet problem.

The Fourier transform of the solution $u(x, y, z, t)$ of our problem

$$w(x, y, z, k) = \sqrt{\frac{k}{\pi}}\, e^{-i\pi/4} \int_{-\infty}^{\infty} e^{ikt} u(x, y, z, t)\, dt$$

satisfies the equation $\Delta w + k^2 w = 0$, the boundary conditions (2), and is the solution of the stationary diffraction problem on S for a plane wave incident from the side $x = -\infty$ of the form e^{ikx} (the Fourier transform of the function $(t-x)_+^{-1/2}$ is equal to $\sqrt{\pi/k}\, e^{i\pi/4+ikx}$). Obviously, the asymptotics of $w(x, y, z, k)$ as $k \to \infty$ is determined by the singularities of $u(x, y, z, t)$ as a function of t for fixed x, y, z. Hence it immediately follows that the first term of the contribution to the asymptotics of $w(x, y, z, k)$ from the spherical wave scattered by the vertex S is equal to $c_2(\omega)e^{ikr}/kr$, where, in order to determine $c_2(\omega)$, one must numerically solve the corresponding Dirichlet problem (we again assume that the direction S does not lie on the boundary of shadow and light).

2. Let us now consider a problem in which it is possible to determine the solution of the nonstationary problem for $r \geq t$.

Let the obstacle S be given by the equations:

$$x = 0, \quad z \leq c|y|, \quad \text{where } |c| \leq 1, \tag{4}$$

i.e. it is a plane screen parallel to the front of the incident wave with an angular cutout, the magnitude of this angle lying within the limits $\frac{1}{2}\pi \leq \alpha \leq \frac{3}{2}\pi$.

We shall show how, for $t > 0$, the solution of our problem is determined at all points x, y, z for which $r = \sqrt{x^2 + y^2 + z^2} \geq t$. At points x, y, z lying outside the t-neighborhood of S, i.e. at a distance from S greater than t, the solution, in view of the finite propagation speed of disturbances for the wave equation equal to unity, will coincide with the solution of the unperturbed problem, i.e. will be equal to $(t - x)_+^{-1/2}$. For points lying at a distance less than t from S, but at a distance greater than t from the edges of S, the solution will coincide with the solution of the problem of reflection of the incident plane wave from the plane $x = 0$. For points lying at a distance less than t from one of the edges of S, but at a distance greater than t from the other edge, the solution will coincide with the solution of the problem of diffraction by a half-plane lying in the plane $x = 0$ and bounded by a straight line containing the corresponding edge. Such a problem is in an obvious way reduced to the two-dimensional problem of diffraction of a plane wave by a half-line. S. L. Sobolev solved a more general problem—the diffraction of a plane wave by an angle (^{1}). In our case the solution of this problem has a very simple form: in the region $\rho < t$, where ρ is the distance to the end of the half-line, the formula holds:

$$u(x, y, t) = \frac{1}{2}\left[(t - x)^{-1/2} - (t + x)^{-1/2}\right].$$

The solution will have the same form in our case as well, but here ρ should be understood as the distance to the corresponding edge.

Thus, each edge S emits a cylindrical wave, within which the solution is known to us. Where these waves intersect, outside the t-neighborhood of the vertex S, the solution is naturally determined as the sum of the contributions made by each of these waves. However, the function constructed in this way is a solution only if the front of the cylindrical wave excited by one of the edges does not intersect, outside the t-neighborhood of the vertex S, another edge, since in that case the cylindrical wave excited by one of the edges would be scattered by the other edge, and it would be necessary to add the corresponding term to the solution. The condition imposed on the magnitude of the angle α: ${}^1/_2\pi \leq \alpha \leq {}^3/_2\pi$, or the equivalent condition for the coefficient c in equations (4), which define S: $|c| \leq 1$, protects us from such a situation.

3. From the considerations given above the following theorems follow. We shall denote: by O_t the sphere with center at the vertex of the angle S and radius t; by B_t, the set of points lying at a distance not greater than t from one of the edges S and at a distance not less than t from the other edge; by C_t, the set of points lying at a distance not greater than t from both edges but not belonging to O_t; by D_t, the set of points lying at a distance less than t from S, but not belonging to O_t, B_t, C_t; finally, by E_t, the remaining points of the space x, y, z. We shall call the Dirichlet problem S the Dirichlet problem for the Laplace equation in the sphere $\rho^2 = x^2 + y^2 + z^2 \leq 1$ with the obstacle S removed and with boundary conditions $u|_S = 0,\ u(1, \omega) = f(\omega)$, where $f(\omega)$ is a given function. We can now formulate the following theorem:

Theorem 2. *The solution of equation (1) with initial and boundary conditions (3) and (2), where* $f(t-x) = (t-x)_+^{-1/2}$, *and* S *is defined by equations (4), has, for* $t > 0$, *the following form:*

Outside the t*-neighborhood of* S*, i.e. in* E_t*:*

$$u(t, x, y, z) = (t - x)_+^{-1/2}.$$

In the t*-neighborhood of* S *outside* O_t*:*

$$u(x, y, z, t) = \begin{cases} (t-x)^{-1/2} - (t+x)^{-1/2} & \text{in } D_t \text{ for } x < 0, \\ 0 & \text{in } D_t \text{ for } x > 0; \\ \dfrac{1}{2}\left[(t-x)^{-1/2} - (t+x)^{-1/2}\right] & \text{in } B_t; \\ -(t-x)^{-1/2} & \text{in } C_t, \text{ if } c > 0, \\ (t-x)^{-1/2} - (t+x)^{-1/2} & \text{in } C_t, \text{ if } c < 0 \text{ and } x > 0, \\ 0 & \text{in } C_t, \text{ if } c < 0 \text{ and } x < 0. \end{cases} \tag{5}$$

In the region $r < t$ the solution has the form:

$$u(t,r,\omega) = [1 + \sqrt{t^2 - r^2}]^{-1/2} w\left(\frac{t - \sqrt{t^2 - r^2}}{r}, \omega\right);$$

$w(\rho,\omega)$ is the solution of the Dirichlet problem S, where $f(\omega)$ is determined by formulas (5), in which one must put $t = 1,\ x^2 + y^2 + z^2 = 1$.

Passing to the stationary diffraction problem, we have:

Theorem 3. *The solution of the stationary diffraction problem for an incident plane wave of the form e^{ikx} on the obstacle S, defined by formulas (4), has the following principal terms of the asymptotics as $k \to \infty$ (for simplicity we restrict ourselves to the region $x > 0$, i.e. the region behind the obstacle):*

A. *For a point x, y, z lying in the shadow region, i.e. for $z < |cy|$, from which it is impossible to drop a perpendicular to either edge:*

$$\frac{e^{i(kr+\pi/2)}}{\sqrt{2}\,kr} \frac{\partial u(1,\omega)}{\partial n},$$

where $r = \sqrt{x^2 + y^2 + z^2}$; ω is the direction of the vector x, y, z; $du(1,\omega)/dn$ is the inward normal derivative at the point $(1,\omega)$ of the solution of the Dirichlet problem S_1, in which the function (5) is taken as the boundary condition for $r = \sqrt{x^2 + y^2 + z^2} = 1$, where one must set $t = 1$.

B. For a point x, y, t lying in the shadow region from which a perpendicular can be dropped to one of the edges:

$$\frac{e^{i(k\rho+\pi/4)}}{\sqrt{2\pi k\rho}} \frac{\cos\psi/2 - \sin\psi/2}{\sin\psi},$$

where ρ is the distance to the corresponding edge, and ψ is the angle, taken with a plus sign, between the perpendicular from the point x, y, z to this edge and the half-line $x > 0,\ y = z = 0$.

C. For a point x, y, z lying in the shadow region from which a perpendicular can be dropped to both edges S, the principal term of the asymptotics consists of two terms of the form (6), corresponding to each of the edges.

D. For a point x, y, z lying in the illuminated region, the principal term of the asymptotics is e^{ikx}.

4. Analogous arguments can be carried out for the diffraction problem on an **arbitrary polyhedral angle**. It is necessary only to take care that the conical wave arising at each of the edges S (the vertex of the cone is the point of intersection of the corresponding edge and the front of the incident wave $x = t$;

the cone is tangent to the sphere O_t with center at the vertex S and radius equal to t) should not intersect, outside the sphere O_t, with other edges.

Let us also note that, from an incident wave of the form $(t-x)_+^{-1/2}$, one can easily pass, by means of the Duhamel integral, to an arbitrary incident wave of the form $f(t-x)$, where for $\xi < 0$, $f(\xi) = 0$.

It is obvious that all the preceding arguments are preserved, after a slight modification, for boundary conditions on S of the form

$$a\frac{du}{dn} + b\frac{du}{dt} = 0,$$

where a and b are constants.

5. Let the obstacle S have the form of a **smooth convex cone** situated in the region $x \geq 0$ and containing the half-axis $x > 0$ (so that no shadow is formed behind the cone). We briefly indicate a way to find the solution of the diffraction problem for an incident plane wave of the form $(t-x)_+^{-1/2}$ in the region $r \geq t$. In this case the following picture of wave fronts obtains: the front of the incident wave $x = t$, the front O_t of the spherical wave $r = \sqrt{x^2+y^2+z^2} = t$, and the front H_t of the wave reflected from the lateral surface of the cone, which intersects S at $x = t$ and is tangent to O_t. To reduce the solution to the Dirichlet problem considered above, it is required to determine the solution in the region between H_t and O_t, i.e. to find the reflected wave. This wave is also an automodel wave and is represented in the form $t^{-1/2}v(r/t, \omega)$, so that for the function $v(\xi, \omega)$ $(\xi = r/t)$ we obtain an equation which is hyperbolic for $\xi > 1$ and degenerates for $\xi = 1$. By any change of variables $\eta = \eta(\xi)$ for which $\eta - 1 = O(\sqrt{\xi - 1})$, this equation is reduced to a nondegenerate one, after which it can be solved by one or another numerical method, for example by means of the ray method (^{2}).

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CITED LITERATURE

[1] S. L. Sobolev, *Transactions of the Seismological Institute of the Academy of Sciences of the USSR*, No. 41 (1934).
[2] V. M. Babich, A. S. Alekseev, *Izvestiya of the Academy of Sciences of the USSR, Geophysical Series*, No. 1 (1958).

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